Platforms as Infrastructures for Mathematics Teachers' Work With Resources

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PREFACE

Textbooks are vital curriculum resources for teaching and learning mathematics in classrooms around the world. They play an integral role in defining mathematics as a school subject, shaping the learning opportunities for students, and supporting teachers' work, learning and professional development. Textbook content, development, and use have been important themes in mathematics education research. Recent years have seen an increasing international interest in textbook research as well as theoretically driven and empirically grounded textbook development. Increasing the understanding of textbooks and their development and of how they are incorporated into teachers' professional work, how they may promote curricular reforms, and how they support students' learning have become important endeavors in the field. Furthermore, the increased availability of digital textbooks entails deep changes in textbook design, content, and use and thus even amplifies the need of understanding the role of textbooks in teaching and learning mathematics.

This volume documents recent issues and the latest development of research on mathematics textbooks presented at the Third International Conference on Mathematics Textbook Research and Development (ICMT3), held at Paderborn University (Germany) from 16 to 19 September 2019. The conference is about all issues related to the development, content and use of mathematics textbooks from kindergarten to university level, in and out of school settings, in paper or digital format. The programme of the conference is structured according to seven themes, which relate to historical perspectives, recent developments as well as future directions:

- **Development of (digital-)textbooks** (e.g. concepts, task design, learning-teaching-trajectories, methodological approaches, quality, design-based research)
- **Content and its presentation in (digital-)textbooks**
- **Use of (digital-)textbooks** (by students, teachers, parents, ...) and related issues (e.g. student achievement, teacher professional development, interaction with other resources)
- **Historical perspectives on textbooks**
- **Comparative studies of (digital-)textbooks**
- **Textbook and policy** (e.g. governmental educational policy, distribution and market strategies)
- **Research on (digital-)textbooks** (e.g. issues, methods, future directions)

4 plenary lectures, 4 symposia, 2 workshops, 40 oral communications, and 13 posters with participants from all the five continents reflect the richness and large scope of research on mathematics textbooks related to these themes. The international programme of the conference was complemented by 2 plenary lectures, 8 lectures and workshops, and 2 round-tables especially for German teachers.

ICMT3 continues the way that has been paved by the International Conference on School Mathematics Textbooks (ICSMT), held in Shanghai (China), in 2011, and proceeded by the (First) International Conference of Mathematics Textbook Research and Development (ICMT-2014), in Southampton (UK) in 2014, and the Second International Conference on Mathematics Textbook Research and Development, that took place in Rio de Janeiro (Brazil) in 2017. The conference-series is going to be continued by the Fourth ICMT in Beijing (China) in 2021.

ICMT3 was organized by an International Programme Committee:

- Sebastian Rezat (Paderborn University, Germany) — Chair
- Lianghuo Fan (East China Normal University, China; University of Southampton, UK) — Co-Chair
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- Michal Yerushalmy (University of Haifa, Israel)

And the Local Organizing Committee from Paderborn University—except Holger Wuschke:

- Mathias Hattermann — Chair
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- Dorothea Backe-Neuwald
- Roland Bender
- Kordula Knapstein
- Sara Malik
- Marcel Sackarendt
- Jan Schumacher
- Kira Starke
- Gerda Werth
- Holger Wuschke (University of Leipzig, Germany)

The proceedings are published prior to the conference. Therefore, the contributions reflect the work that was done in preparation for the conference based on a peer-review process among the contributors of the conference. We would like to thank all reviewers for their valuable work that helped to improve the quality of all papers in this volume.

We further thank the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), East China Normal University Press, Klett Publishing Company, and Paderborn University for supporting and funding the conference.

August 2019

Sebastian Rezat

Lianghuo Fan

Mathias Hattermann

Jan Schumacher

Holger Wuschke

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Part I.

Plenaries

REVISITING RESOURCES AS A THEME IN MATHEMATICS (TEACHER) EDUCATION

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Almost twenty years ago, I offered a reconceptualization of resources as a theme in mathematics teacher education (Adler, 1998, 2000). The reconceptualization had three dimensions to it. First was the suggestion that we needed to shift from viewing 'resource' only as a noun, and consider it also as a verb. This shift brings into view teachers-working-with-resources, or simply resources-in-use; and more openness in understanding how teachers re-sourced their practice as they enacted, for example, curriculum or pedagogic change. Secondly, the conceptualization of the notion of resources extended beyond the material and physical (e.g. textbooks, texts, chalkboards, artefacts) to include socio-cultural resources like language and time. As less visible, but nevertheless means to enabling teaching and learning, these too should be considered as further resources (or obstacles?) in school mathematics practices. Third was a theoretical orientation, discussed below, to resources-in-use informed by an understanding of school mathematics as a 'hybrid' practice, and of the accessibility of resources being a function of their 'transparency'.

This reconceptualization emerged from a research and development teacher professional development project in post-apartheid South Africa, with participating teachers coming from schools serving learners in low income communities. In this context, the availability of and access to resources, while being addressed by the post-apartheid state, was still severely limited. In addition, curriculum reform was underway, with advocacy for learner centered pedagogic practices. This entailed, on the one hand, devolving autonomy for learning to learners' activity; and on the other, in mathematics, connecting mathematical activity to learners' everyday realities, resulting in what I referred to as "hybridization" – enacted practices emerging across these continua. I argued then that texts (e.g. worksheets with activities) and artefacts (e.g. geoboards) while produced with mathematical intentions, did not have mathematics "shining through" them. In Lave & Wenger's (1991) terms, these resources needed to be "transparent", visible so that they could be used, and simultaneously invisible, enabling access to mathematics. School mathematics required mediation, and critically so if meaning-making was to be through learner activity with 'new' material resources, and contextualized in practices other than mathematics.

At the same time – and as a multilingual country – there was focused work shifting the discourse away from learner's main languages being considered "a problem", to their languages being "a resource" for learning and teaching (e.g. Adler, 2001; Setati, 2005). Here too were arguments for the importance of 'transparency', where

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deliberate attention to language (making language visible) was in tension with language simultaneously needing to be invisible, providing access to mathematics (Adler, 1999).

In the past two decades there has been extensive research and development on resources in mathematics education, focused, for example, on teachers' use of curriculum materials (Remillard, Eisenmann, & Lloyd, 2009) their documentation practices and on resources as "lived" (Gueudet, Pepin, & Trouche, 2012) . Renewed interest in research textbooks in mathematics education was manifest in two special issues of ZDM (2013 and 2018), and studies of the teacher-textbook relation (e.g. Leshota & Adler, 2018). In addition, there are a number of reviews of this accumulating work on resources and textbooks in edited book volumes (Fan, Trouche, Chunxia, Rezat, & Visnovska, 2018; Styilianides, 2016; Trouche, Gueudet, & Pepin, forthcoming). And as Fan & Schubring (2018) show, research related to mathematics textbooks has a far longer history.

In this presentation I will draw from this extensive literature and reviews of the field to ask:

What have been the developments related to resources in mathematics education, empirically, methodologically and theoretically since the early 2000s? How does this wider range of research on resources relate to focused work on textbooks, particularly in the current era of proliferating electronic resources? Where and how have these developments emerged and evolved? This retrospective will provide the landscape for revisiting the conceptualisation of resources previously offered and for critical reflection on its current salience, and with textbooks in view. My initial work suggests a number of issues come to the fore. These will form the substance of my presentation at the ICMT-3 conference.

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References

Adler, J. (1998). Resources as a verb: Recontextualising resources in mathematics education In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 1–18). Stellenbosch: University of Stellenbosch, Faculty of Education.

Adler, J. (1999). The dilemma of transparency: Seeing and seeing through talk in the mathematics classroom. *Journal for Research in Mathematics Education*, *30*(1), 47–64.

Adler, J. (2000). Conceptualising resources as a theme for mathematics teacher education. *The Journal of Mathematics Teacher Education*, *3*(3), 205–224.

Adler, J. (2001). *Teaching mathematics in multilingual classrooms*. Dordrecht: Kluwer Academic Publishers.

Fan, L., & Schubring, G. (2018). Recent advances in mathematics textbook research and development: An overview. *ZDM Mathematics Education*, *50*, 765–771. https://doi.org/10.1007/s11858-018-0979-4

Fan, L., Trouche, L., Chunxia, Q., Rezat, S., & Visnovska, J. (Eds.). (2018). *Research on mathematics textbooks and teachers' resources*. Cham, Switzerland: Springer.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.). (2012). *From text to 'lived' resources: Mathematics curriculum materials and teacher development.* Dordrecht: Springer.

Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation.* Cambridge: Cambridge University Press.

Leshota, M., & Adler, J. (2018). Disaggregating a mathematics teachers' pedagogic design capacity. In L. Fan, L. Trouche, Q. Chunxia, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources* (pp. 89–118). Cham: Springer.

Remillard, J. T., Eisenmann, B. A. H., & Lloyd, G. M. (Eds.). (2009). *Mathematics teachers at work connecting curriculum materials and classroom instruction.* UK: Routledge.

Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education*, *36*(5), 447–466.

Styilianides, G. (Ed.) (2016). *Curricular resources and classroom use: The case of mathematics.* Oxford: Oxford University Press.

Trouche, L., Gueudet, G., & Pepin, B. (Eds.). (forthcoming). *The 'resource' approach to mathematics education.* Cham: Springer.

MATHEMATICAL PROBLEM POSING, CURRICULUM DEVELOPMENT, AND PROFESSIONAL LEARNING

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Problem posing, the process of formulating and expressing problems based on a given situation, is an essential practice in mathematics and other disciplines (Cai & Hwang, 2019; Cai et al., 2015; Silver, 1994; Singer, Ellerton, & Cai, 2015). This is acknowledged in policy documents for school mathematics. For example, thirty years ago, the National Council of Teachers of Mathematics (NCTM) published what was to become a widely influential standards document, the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). This document contained NCTM's first formal recommendation that students should "have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138).

Similarly, in China, one of the objectives of the curriculum reforms for 9-year compulsory education is for students to learn how to pose problems from mathematical perspectives, learn how to understand problems, and learn how to apply their knowledge and skills to solve problems so as to increase their awareness of mathematical applications (Chinese Ministry of Education, 2011). The secondary school mathematics curriculum is intended to enhance students' abilities to pose, analyze, and solve problems from mathematical perspectives.

Yet, curriculum materials widely used in China and the United States fail to incorporate problem posing in a substantial and consistent way. For example, Cai and Jiang (2017) compared the inclusion of mathematical problem posing in popular Chinese and U.S. elementary mathematics textbooks (specifically, the textbook series published by People's Education Press in China and Investigations in Number, Data, and Space and Everyday Mathematics in the United States).

Table 1 shows both the total number of tasks Cai and Jiang identified in each of the Chinese and U.S. elementary mathematics textbooks as well as the percentage of those tasks that could be classified as problem-posing tasks. They found only a small proportion of problem-posing activities in any of the curricula, and the proportion fluctuated across grade levels. Cai and Jiang (2017) also examined the mathematical content areas in which the problem-posing tasks were found. For all three curricula, the vast majority of the problem-posing activities (over 90% for the U.S. textbooks and nearly 80% for the Chinese textbooks) were related to number and operations. Only a few problem-posing activities were situated in the content areas of algebra, geometry, measurement, and data analysis. Even though the number and operations content area tends to occupy the most space in elementary mathematics textbooks, this distribution

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of problem-posing tasks remains disproportional, and it reflects a haphazard approach to including problem posing in the mathematics curriculum.

Grade	Chinese		U.S.			
	People's Education Press		*Investigations*		*Everyday Mathematics*	
	n	% PP	*n*	% PP	*n*	% PP
1	669	3.74	490	0	-- [2]	--
2	711	5.06	741	1.62	1,651	1.03
3	694	5.62	832	0.72	1,322	1.06
4	699	1.57	760	1.97	1,565	1.28
5	821	2.80	726	2.62	1,896	1.16
6	745	2.01	-- [1]	--	1,673	0.42
Total	4,339	3.43	3,549	1.47	8,107	0.99

[1] *Investigations* does not have a Grade 6 textbook.

[2] *The data for Everyday Mathematics Grades 1 and 2 was combined because there is only one combined Student Reference Book for those the two grades.*

Table 1. Total Number of Tasks and Percentages of Problem-Posing (PP) Tasks in the Chinese and U.S. Mathematics Textbooks Series from Grades 1 to 6

Because current curriculum materials do not incorporate significant and consistent experiences with problem posing for students, it is unreasonable to expect that problem posing will spontaneously be given much attention in classrooms (Lloyd et al., 2017). Teachers already face multiple demands on their time and attention. On any given day, they cannot devote large amounts of time preparing for significant changes in their upcoming lessons. Thus, to gain buy-in from teachers (and students), any strategy to integrate problem posing more effectively in mathematics classrooms should avoid being burdensome or perceived as a radical change in practice that would require a lot of time to adapt to. Instead, integrating problem posing should build on existing, common practices.

In that vein, I propose three recommendations to better integrate problem posing into the school mathematics curriculum: 1) empowering teachers as curriculum redesigners to reshape existing curriculum materials in simple ways that create learning opportunities for mathematical problem posing; 2) enhancing existing curricula with additional problem-posing tasks that include support in the form of sample posed problems; and 3) encouraging students to pose problems at different levels of complexity. These three recommendations are practical and feasible based on evidence from teacher professional development focused on problem posing to teach mathematics (Cai et al., 2019). In fact, we are conducting a longitudinal study to investigate how teachers learn to teach mathematics using problem posing, and then their teaching impact on students' learning (Cai et al., 2019).

Through teacher learning, teachers increase their knowledge and change their beliefs, and then change their classroom instruction aiming to improve students' learning. In the research project, the problem-posing workshops have been designed and focused on changing teachers' beliefs and increasing their knowledge of problem posing and teaching mathematics through problem posing. The problem-posing workshop was designed and framed in the context of effective teacher professional development. Although the field of mathematics education knows little about how to support teachers to become better problem posers and teach mathematics through problem posing, substantial evidence has emerged concerning the features of PD that have a positive impact on teachers' instructional practice and students' learning (e.g., Vescio et al., 2008). These features include: (a) a focus on content, (b) building on student learning and thinking, (c) close alignment with practice, (d) building a learning community, and (e) PD that is ongoing.

We found that although many teachers may have little experience with using problem posing activities in the mathematics classroom, problem posing offers enticing benefits in the potential for deepening students' engagement with mathematics and gaining a better understanding of students' mathematical thinking. The relatively minimal investment in professional development that would be needed to help teachers gain confidence in using problem posing in mathematics instruction would be well spent and can see effect (Cai et al., 2019).

References

Cai, J., & Hwang, S. (2019). Learning to teach mathematics through problem posing: Theoretical considerations, methodology, and directions for future research. *International Journal of Educational Research.* Online First. https://doi.org/10.1016/j.ijer.2019.01.001.

Cai, J., & Jiang, C. (2017). An analysis of problem-posing tasks in Chinese and US elementary mathematics textbooks. *International Journal of Science and Mathematics Education, 15*(8), 1521–1540.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. M. Singer, N. Ellerton, & J. Cai (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 3–34). New York, NY: Springer.

Cai, J., Chen, T., Li, X., Xu, R., Zhang S., Hu, Y., Zhang, L., & Song, N. (2019). Exploring the impact of a problem-posing workshop on elementary school mathematics teachers' problem posing and lesson design. *International Journal of Educational Research.* Online first. https://doi.org/10.1016/j.ijer.2019.02.004.

Chinese Ministry of Education. (2011). *Mathematics curriculum standard of compulsory education* (2011 version). Beijing, China: Beijing Normal University Press.

Lloyd, G. M., Cai, J., & Tarr, J. E. (2017). Issues in curriculum studies: Evidence-based insights and future directions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 824–852). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.

Singer, F. M., Ellerton, N. F., & Cai, J. (Eds.). (2015). Mathematical problem posing: From research to effective practice. New York, NY: Springer.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. *Teacher and Teaching Evaluation, 24*(1), 80–91.

MATHEMATICS (E-)TEXTBOOKS: HELP OR HINDRANCE FOR INNOVATION?

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Textbooks may help education innovation as they can support teachers to enact renewed curriculum intentions in classroom processes. At the same time textbooks may also hinder real innovation as they limit teachers' opportunities to (re)design the curriculum and develop curriculum design capacity (e.g., Pepin, Gueudet, & Trouche, 2017), by an overdose of detailed scripts that reduce teachers to 'technical slaves'. E-textbooks are heralded to be interactive and to support teachers in their everyday classroom practices, as well as in their curriculum (re-)design, through innovative and collaborative work with colleagues. However, access to useful subject-didactical resources does not always lead to curriculum renewal and innovative practices.

In this presentation I use the theoretical notion of *connectivity* (e.g., Pepin et al., 2016; Gueudet et al., 2018) in association with *Documentational Approach to Didactics* (e.g., Trouche, Gueudet, & Pepin, 2019), and selected concepts from the field of curriculum theory, to propose in which ways e-textbooks can be used, and what needs to be considered for their use, to further mathematics teacher professional development, to lead to curriculum renewal by teachers and innovative practices.

In our current 21st century environment we are surrounded by technology; we cannot disconnect from technology, and this is true for textbooks too (e.g., Pepin, Choppin, Ruthven, & Sinclair, 2017). Textbooks still play a vital role, albeit they are nowadays often of a different nature – they are e-textbooks. In earlier works (e.g., Pepin et al., 2016) we have defined e-textbooks as "an evolving structured set of digital resources, dedicated to teaching, initially designed by different types of authors, but open for re-design by teachers, both individually and collectively" (p.644). These changes and developments, from static books to "dynamic" e-textbook, call for different teacher competences, and these are related to the design of teaching, and of the curriculum, with digital resources. With the change of the nature of textbooks, we need new ways of supporting teachers with their work, and this is likely to include different ways of studying their work: how can we support teachers in their endeavour of developing a coherent curriculum for their students? How can we help teachers to select suitable resources for teaching and learning the mathematics at hand?

I have chosen to introduce and use the notion of *connectivity* as a critical feature of curriculum coherence, which can be used -beyond the case of e-textbooks analysis- for helping mathematics teachers to connect the different components of the mathematics curriculum in time of digitalization. This choice allows me to develop a frame for developing learning/teaching trajectories (with the help of e-textbooks) for innovative teaching in a digital age. Considering the mathematics curriculum in different

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representations: intended, enacted, attained; and at different curriculum levels: nano - student, micro - classroom, meso - school, macro - national level (Thijs & Van den Akker, 2009), mathematics teachers can be supported/guided by the e-textbook to align their goals with the affordances of the connected resources.

What I argue in this presentation is that this kind of support needs to be deliberately and systematically addressed, in order to help teachers to develop a more coherent overview of the mathematics curriculum and its didactically sensitive/suitable tools and resources, in order to realize their mathematical and pedagogical goals in the classroom. Teachers need support to make such connections across the different components of the curriculum (for internal coherence) and between the different curriculum levels (for overall coherence). Educative curriculum materials (Krajcik & Delen, 2017), in digital format (Pepin, 2018), are likely to be helpful to provide such support. Such materials, connected to e-textbooks, may focus on a limited number of essential characteristics for curriculum renewal. They may

- help teachers to orientate on and practice with new elements in their teaching repertoire;
- guide teachers in role taking experiences that exemplify new pedagogical approaches;
- create opportunities for shared reflection by teachers that may challenge also their beliefs about appropriate teaching.

Based on such collaborative, specific experiences, teams of teachers are likely to be stimulated and supported to redesign their overall mathematics education approaches.

Such educative materials are best designed and piloted by small teams of mixed composition: teachers, teacher educators, curriculum designers, and researchers. The development process is said to be iterative, with gradually shifting emphasis in quality criteria: from relevance, to consistency, practicality, and effectiveness.

The concluding argument is that beside appropriate (e-)resources teachers need support in professional development collectives, to develop an awareness of connectivity and a capacity to increase internal and external curriculum coherence.

References

Gueudet, G., Pepin, B., Sabra, H., Restrepo, A., & Trouche, L. (2018). E-textbooks and connectivity: proposing an analytical framework. *International Journal for Science and Mathematics Education*, *16*(3), 539–558.

Krajcik, J., & Delen, I. (2017). The benefits and limitations of educative curriculum materials. *Journal of Science Teacher Education, 28*(1), 1–10.

Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: foundations for change. *ZDM Mathematics Education, 49*(5), 645–661.

Pepin, B., Gueudet, G. & Trouche, L. (2017). Refining *teacher design capacity:* Mathematics teachers' interactions with digital curriculum resources. *ZDM Mathematics Education*, *49*(5), 799–812. https://doi.org/10.1007/s11858-017-0870-8

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. (2016). E-textbooks in/for teaching and learning mathematics: A potentially transformative educational technology. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 636–661). New York: Taylor & Francis.

Pepin, B. (2018). Enhancing teacher learning with curriculum resources. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.). *Research on mathematics textbooks and teachers' resources: Advances and issues,* Springer (pp. 359–374). Cham, Switzerland: Springer.

Thijs, A., & van den Akker, J. (2009). *Curriculum in development.* Enschede, The Netherlands: Netherlands Institute for Curriculum Development (SLO).

Trouche, L., Gueudet, G., & Pepin, B. (Eds.). (2019). *The 'Resource' approach to mathematics education*. Cham: Springer.

ENHANCING AND UNDERSTANDING STUDENTS' PROCESSES OF MATHEMATIZATION AND ACTIVE KNOWLEDGE ORGANIZATION — DIDACTICAL DESIGN RESEARCH FOR AND WITH TEXTBOOKS IN THE KOSIMA-PROJECT

Susanne Prediger

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INTRODUCTION

In the Kosima Project, 16 years were spent on enhancing and understanding students' learning processes, with a specific focus on constructing meanings by mathematization and active knowledge organization. Initiating rich processes of mathematization is a central aim for all mathematics education. However, many obstacles appear for these processes to generate solid and sustainable mathematical knowledge. For overcoming these obstacles, our textbook research was conducted as Didactical Design Research, which systematically combined two aims: (a) research-based design of teaching-learning arrangements and (b) topic-specific design-based research for a deeper understanding of the initiated learning processes. Finally, a field trial could prove effectiveness of the design.

PROJECT ARCHITECTURE: DESIGN RESEARCH

The talk will report on the long-term design research project KOSIMA (2005-2020, cf. Hußmann et al., 2011; Barzel et al., 2013). It follows the methodology of Didactical Design Research (Gravemeijer & Cobb, 2006) with its dual aim of *designing teaching-learning-arrangements* for a complete middle school curriculum (Grades 5 to 10) and *empirically researching* the teaching-learning-processes and their conditions. The developed curriculum has been published as the textbook *Mathewerkstatt* (Barzel et al., 2012-2017) and a comprehensive teachers' manual.

The collaborative project involved the design team (including 22 experienced reflective practitioners and the 4 project leaders), researchers (4 project leaders and 13 PhD students), the commercial publisher (with 2-4 copy editors), and project teachers (experimenting with the designs in their mathematics classrooms).

All teaching-learning-arrangements of the curriculum were developed in iterative cycles of design, evaluation (by expert discussions and classroom experiments), and redesign. Whereas the design and evaluation steps of the project referred to the entire textbook, the deeper research was organized in several smaller design research studies that necessarily had to address more narrow research questions for topic-specific aspects. These studies used different concrete research methods and designs (e.g., quasi-experimental controlled trials or design experiments in laboratory settings with up to four cycles, e.g., Leuders & Philipp, 2012; Prediger & Schnell, 2014). Empirical

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evidence of summative effectiveness was provided in a quasi-experimental field trial with pre-post-test and control group.

DESIGN PRINCIPLES FOR THE CURRICULUM AND MATERIALS

The design of the Kosima curriculum followed design principles which were partly borrowed from Realistic Mathematics Education (Freudenthal, 1991) and elaborated with respect to developing conceptual understanding (Hiebert & Carpenter, 1992, Prediger & Schnell, 2014) and active knowledge organization (Barzel et al., 2013):

- support students' sense-making processes
- conceptual understanding before procedures by collaborative meaning construction
- guided reinvention and mathematizations in cognitively demanding explorations
- support active knowledge organization in scaffolded vertical mathematization
- connect multiple representations and diverse mathematical approaches

For realizing these design principles for different mathematical topics, specifying and structuring the learning contents turned out to be a crucial working area. This involves epistemological as well as empirical analysis in order to best align students' realized learning pathways to the intended learning trajectories (Hußmann & Prediger, 2016). Initiating rich processes of mathematization is a central aim for all mathematics education. However, many obstacles appear for these processes to generate solid and sustainable mathematical knowledge. So, the iterative design cycles led to elaborating task formats which can scaffold teachers' teaching as well as students' learning.

References

Barzel, B., Hußmann, S., Leuders, T., & Prediger, S. (Eds.). (2012–17). *Mathewerkstatt 5–10.* Berlin: Cornelsen.

Barzel, B., Leuders, T., Prediger, S., & Hußmann, S. (2013). Designing tasks for engaging students in active knowledge organization. In A. Watson et al. (Eds.), *ICMI Study 22 on Task Design—Proceedings of the Study Conference* (pp. 285–294). Oxford: ICMI.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer.

Gravemeijer, K., & Cobb, P. (2006). Design research from the learning design perspective. In J. van den Akker, K. Gravemeijer , S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 17–51). London: Routledge.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Hußmann, S., Leuders, T., Prediger, S., & Barzel, B. (2011). Kontexte für sinnstiftendes Mathematiklernen (KOSIMA) – ein fachdidaktisches Forschungs- und Entwicklungsprojekt. *Beiträge zum Mathematikunterricht*, 419–422.

Hußmann, S., & Prediger, S. (2016). Specifying and structuring mathematical topics. *Journal für Mathematik-Didaktik, 37*(Suppl. 1), 33–67.

Prediger, S., & Schnell, S. (2014). Investigating the dynamics of stochastic learning processes: A didactical research perspective, its methodological and theoretical framework, illustrated for the case of the short term–long term distinction. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic Thinking* (pp. 533–558). Dordrecht: Springer.

Part II.

Symposia

Symposium A: Research potential of interactive textbooks: New perspectives for research in mathematics education

RESEARCH POTENTIAL OF INTERACTIVE TEXTBOOKS: NEW PERSPECTIVES FOR RESEARCH IN MATHEMATICS EDUCATION

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Interactive textbooks as digital curriculum resources (Pepin et al., 2017) offer unique ways of implementing digital media into mathematics classrooms. This symposium is looking out to frame questions arising from these rich possibilities for research in mathematics education, regarding questions about students' use of interactive features, the effect of those features on students' outcome, as well as the utilization of process data or log data as unobtrusive measurements of students' behaviour.

PURPOSE AND STRUCTURE OF THE SYMPOSIUM

There is an ongoing discussion about how and whether digital media should be used in schools to teach mathematics. Typically, questions asked concern the motivational or cognitive development of students when educational technology is used. However, the digital devices may also be used to gain information on students' learning. In this symposium, we want to discuss the research potential of digital media with a focus on interactive textbooks.

Interactive Textbooks Offering New Modes of Instruction

The use of digital and interactive textbooks allows for other modes of instruction than paper-based material (e.g., Pepin, Choppin, Ruthven, & Sinclair, 2017). For example, different ways of adaptive choice of task difficulties (e.g., Leutner, 2004) and various types of feedback (e.g., Hattie & Timperley, 2007) can be investigated to see what suits students the most—and whether one way fits all. Moreover, hands-on activities which can be implemented in digital textbooks on touchscreen devices can be thought of as holistic ways of interaction between students and textbooks, opening up new questions about dynamic explorations, the role of semiotic mediation (Mariotti, 2009).

Within this symposium, Maximilian Pohl and Florian Schacht will give a structural analysis of features implemented in interactive textbooks (e.g., options for dynamic explorations), guided by a qualitative study of how fifth-grade students use those features to learn mathematics. Furthermore, Sebastian Rezat will present findings from a quantitative analysis on the effect of feedback on primary students' answers during classroom instruction, leading to the result that the implementation of different kinds of feedback does not seem to be beneficial for primary school students' math achievement.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 37–38). Paderborn: Universitätsbibliothek Paderborn.

Interactive Textbooks as Research Tools

We want to discuss newly developed methods to use digital textbooks as a research tool (e.g., Hoch, Reinhold, Werner, Richter-Gebert, & Reiss, 2018). Capturing students' interactions with an interactive textbook opens up new ways to assess students' mathematical knowledge—not only at the end of the development of mathematical concepts in standardized tests, but during the development of mathematical concepts in real classroom scenarios. For example, process data from interactive textbooks in form of log files (e.g., Goldhammer, Naumann, Rölke, Stelter, & Tóth, 2017) allow to gain insights into students thinking and the comfortable assessment of variables like time on task.

Within this symposium, Frank Reinhold, Anselm Strohmaier, Stefan Hoch, and Kristina Reiss will present findings of a quantitative analysis of six-graders engagement during mathematics instruction, showing that process data can be utilized as an unobtrusive measure of engagement.

Addressees of this Symposium

This symposium reaches out to both researchers and educators who either develop interactive mathematics textbooks, use interactive textbooks to alter the cognitive or motivational development of students during mathematics education, or use interactive textbooks to assess mathematics utilizing students' log data.

References

Goldhammer, F., Naumann, J., Rölke, H., Stelter, A., & Tóth, K. (2017). Relating product data to process data from computer-based competency assessment. In D. Leutner, J. Fleischer, J. Grünkorn, & E. Klieme (Eds.), *Competence assessment in education* (pp. 407–425). https://doi.org/10.1007/978-3-319-50030-0_24

Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research*, *77*(1), 81–112. https://doi.org/10.3102/003465430298487

Hoch, S., Reinhold, F., Werner, B., Richter-Gebert, J., & Reiss, K. (2018). Design and research potential of interactive textbooks: the case of fractions. *ZDM Mathematics Education*, *50*(5), 839–848. https://doi.org/10.1007/s11858-018-0971-z

Leutner, D. (2004). Instructional design principles for adaptivity in open learning environments. In N. M. Seel & S. Dijkstra (Eds.), *Curriculum, plans, and processes in instructional design. International perspectives.* (pp. 289–308). Mahwah, N.J.: Erlbaum.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM Mathematics Education*, *41*(4), 427–440. https://doi.org/10.1007/s11858-009-0199-z

Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: foundations for change. *ZDM Mathematics Education*, *49*(5), 645–661. https://doi.org/10.1007/s11858-017-0879-z

HOW DO STUDENTS USE DIGITAL TEXTBOOKS?

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With the transition from printed to digital textbooks the hope of new forms of textbook concepts is coexistent. On the one hand, this means that due to the digital nature new structural features of textbooks can be realised, but on the other hand, the effects on the learning of mathematics and on the use of textbook elements by the learner is of great research importance. Therefore, this contribution addresses a structural analysis of digital mathematics textbooks as well as an empirical study of student uses of textbook elements with the aim of identifying usages of textbook elements for the learning of mathematics.

INTRODUCTION

Textbooks are generally considered to play a dominant role in the teaching and learning of any subject, but especially in mathematics (cf. Fan et al., 2013, p. 635–636). This is due to their unique role as a "conveyor of the curriculum (…) where they "serve as resources which introduce readers to worlds which are not immediately obvious or cannot be experienced directly. In particular, textbooks have their power in providing an 'organized sequence of ideas and information' to structured teaching and learning" (Fan et al., 2013, p. 635). Therefore, the dominant role of the resource *textbook* is constituted by the (mathematical) content structured in sequences (age-group levels, chapters, learning units). Regarding digital textbooks, technological potentialities may affect both content and structure of textbooks as well as the relationships between textbook authors, teachers and learners due to the possibility for teachers on collectively working together and for students on communicating with each other as well as with teachers during their textbook work (cf. Pepin et al., 2016, p. 639).

In Germany, first digital textbooks have been published in the last years demonstrating the broad variety of digital textbook concepts varying from simple adaptations of the printed versions – very often with the same structure and look but only in a digitalised format – to new textbook concepts ranging from online platforms to electronic textbook versions. On the one hand, this broad variety of digital mathematics textbooks goes along with structural changes as the idea of 'textbook' as a book is loosened and, therefore, creates new textbook concepts. But although the effects on the structure of mathematics textbooks are a decisive research focus, targeting the ramifications of the content and the actual use of digital textbooks by students is of equivalent importance. Thus, this paper focuses both on the structural changes as well as on the actual student use of digital textbooks.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 39–44). Paderborn: Universitätsbibliothek Paderborn.

THEORETICAL BACKGROUND

When working with textbooks, the *user* interacts with the *textbook* as well as with the content, i.e. the *mathematics*. Hereby, the textbook acts as a mediator of the mathematical content. All three aspects interact, correlate and influence each other; thus, we conceptualize the use of (digital) textbooks within the socio-didactical triangle (cf. Rezat, 2009, p. 24) as the interaction between *textbook – student – mathematics*. In this way, we can not only draw inferences about the textbook, the individual student or the mathematical content, but especially about the correlation of all three aspects.

In order to discuss the textbook elements within the student use of digital textbooks, a qualitative content analysis (cf. Mayring, 2008) of the digital textbook on the micro level was carried out in advance, i.e. concerning the "structure of individual thematic sections" (Rezat, 2008, p. 48). The textbook elements determined during this process and, therefore, construct the textbook's learning unit will be referred to (based on Rezat, 2009) as 'structural elements' in the following. At the descriptive level, we first identified the structural elements of digital textbooks according to their characteristics. In contrast to the structural elements for traditional printed textbooks, the structural elements for digital textbooks were analysed for their digital or analogue nature in the first place meaning that they were examined regarding their underlying digital characteristics (e.g. *drag and drop* mode, dynamic visualisations, possibility of direct feedback, etc.). Overall, it can be shown that, in addition to the structural elements from traditional textbooks (e.g. boxes containing basic knowledge, texts, etc.), exercises are realized in different dynamic formats (exercises with a *drag and drop* mode, calculation or note character as well as animations and explorations) and that several possibilities for checking your results are given (displaying the solution, showing the solution path, dynamic verification of the results) (cf. Pohl & Schacht, 2018). Due to that, we also speak of 'digital' structural elements when referring to the 'new' structural elements.

Concerning research results on student uses of traditional textbooks, Rezat (2011) ascertained that students mostly engage in the activities of *practicing* and *solving tasks and problems* and that they usually work with *exercises* and *boxes containing basic knowledge* (cf. Rezat, 2011, p. 171). For digital textbook uses, it will be relevant to see how students actually use the determined 'digital' structural elements, which is the focus for the empirical research. In order to approach this research question, looking at signs (i.e. language as well as gestures) the users produce when working with the textbook provides access to the students' textbook practices. This is described by the theory of *semiotic mediation*:

> "[W]ithin the social use of artifacts in the accomplishment of a task (…) shared signs are generated. On the one hand, these signs are related (…) to the artifact used, and, on the other hand, they may be related to the content that is to be mediated" (Bartolini Bussi & Mariotti, 2008, p. 752).

Accordingly, this theory helps to differentiate between signs directed towards the artifact (*artifact signs*) and the mathematical content (*mathematics signs*) and, therefore, provides a basis for the analysis of the student textbook uses. Besides the two categories of signs, a third category (*pivot signs*) addresses “both (…) the activity with the artifact (…) and to the mathematical domain” (Bartolini Bussi & Mariotti, 2008, p. 757) as their “meaning is related to the context of the artifact but assumes a generality through its use in the natural language” (757). In this contribution, the theory of *semiotic mediation* serves as an evaluation method in order to identify student uses and to illustrate the process of using the textbook for the learning of mathematics. Hereby, we focus on the production of signs by the students when working with the digital textbook in the context of learning mathematics. By doing that, the research question “How do students use structural elements of digital textbooks?” will be discussed.

RESEARCH DESIGN

For the identification of structural elements, the chapters "Area and Perimeter" as well as "Negative Numbers" of the digital textbook "Brockhaus Lehrwerke: Mathematik 5. Klasse" (Hornisch et al., 2017) were first examined for their technological characteristics. On the empirical level, against the background of the descriptive analysis of textbooks, it will then be examined in more detail how learners explicitly work with 'digital' structural elements.

In order to investigate this research question, six pupils from the fifth grade of a secondary school worked in groups of three on a total of five sessions (60–90 minutes) on school-internal iPads on the above-mentioned topics with the digital textbook. Clinical interviews were conducted and the students’ textbook work was videotaped on two video cameras; in addition, the screens of the iPads were recorded with the screen recording function of the iPad. The students worked in different situational conditions (i.e. individual, partner and group work) and on given as well as individual assignments. In addition, they could make notes, calculations, etc. with pen and paper at any time and move freely within the digital textbook.

RESEARCH RESULTS

In order to address and answer the empirical research question above, we will show the use of structural elements by two examples. The first transcript (table 1) focuses on the selection of different structural elements by a student for justifying a mathematical proposition while the second transcript (table 2) shows how the student discovers new solutions and ways of performing the task through the explorative use or rather due to the digital nature of the structural element.

The student in the first example was asked to find a way to calculate the area of squares. Prior, the student had worked with the textbook on several structural elements (texts, boxes containing basic knowledge, calculation exercises and dynamic explorations) on finding the area of rectangles. He then was asked to write down his idea of how to calculate the area of squares in the exercise (cf. figure 1):

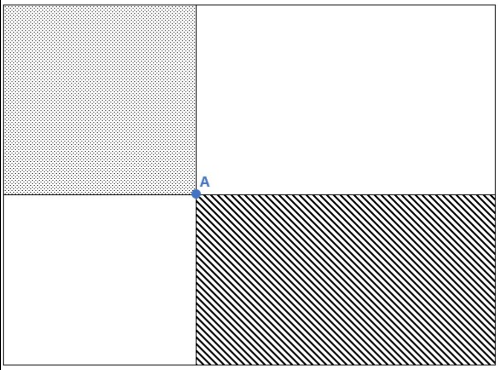
Actually, exactly the same. (...) Actually the same, because here [*scrolls in the textbook to the structural element "exploration"*] at this exercise with the rectangles you could also ... here! It was even a square at the beginning. (...) a times b = A. That was exactly the same formula as here with the rectangle [*scrolls in the textbook to the structural element "box containing basic knowledge"*]. (...) I'll take a look at the solution. It also states that this is exactly the same.

Table 1. Transcript showing the student's selection of different structural elements for reasoning his previously made proposition

Using the theory of *semiotic mediation* and, therefore, having a look at the *artifact signs* first, we can see that the student refers to the artifact every time he selects specific structural elements (T. 1–2: "here [*scrolls in the textbook to the structural element "exploration"*] at this exercise", T. 3: "here!", T. 4–5: "here with the rectangle [*scrolls in the textbook to the structural element "box containing basic knowledge"*], T. 5: "I'll take a look at the solution."). As a second step, the student does not only refer solely to the artifact itself, but also indicates an interpretation of the mathematical content at the same time (*pivot signs*). This can be seen in turns 2–4 where he does not only point to structural elements within the textbook (T. 2: "at this exercise with the rectangles", T. 4: "here with the rectangle"), but also explains the representation of the mathematical content given by the textbook (T. 3: "It was even a square at the beginning.", T. 3–4: "a times b = A. That was exactly the same formula."). Having a look at the *mathematics signs*, the student argues altogether that calculating the area of squares and rectangles is "exactly the same" (T. 6). Here, the student does not refer to the textbook anymore but concludes that the area of squares and rectangles is calculated in the same way and, therefore, argues on a mathematical basis. Altogether, this example illustrates how the student uses the textbook in the context of learning mathematics as he justifies his statement from the beginning (T. 1) by deliberately selecting specific structural elements supporting his mathematical assertion.

In contrast to the first example, the second transcript demonstrates a learning process where different usage-related levels can be identified in the student's examination of the structural element. The task was to find places of the point A where the area of the dotted and shaded rectangles is the same. When working on the task, the student initially did not realize that the position of point A could be changed dynamically and thus counted by reference to the static image how many boxes point A had to be moved to the right so that the two rectangles have the same area. By doing that, he found one solution (*10x15 or 15x10*).

When using the interactive graphic (table 2), the student thus verifies his previously assumed solution with the help of the dynamic visualization in order to obtain a solution by the digital textbook (T. 3–4: "Ehm ... [*moves point A in the interactive graphic*] So ... fifteen. So that's the same with both.") (*artefact sign*). His action,



Task: Are there positions of the point A where the areas of the two highlighted rectangles are the same size?

I cannot find any more [solutions] (...) Here you can do that too [*sees the interactive graphic*]. (...) Ehm ... [*moves point A in the interactive graphic*] So ... fifteen. So that's the same with both. (...) But you can also change the [*drags point A vertically*] ehm depth (...) [*moves point A now vertically and horizontally*] (...) So I think that there is also a point where they both do not look the same. (...) [W]ould look something like that, where you cannot see at first glance that they are the same size, but that the number of boxes inside is the same.

Table 2. Textbook task (cf. Hornisch et al., 2017) and the transcript of the student showing the use of a 'digital' structural element

therefore, relates particularly to the dynamic advantage of the structural element of the digital textbook and thus can be categorized as 'structure element-related usage' (1). As a second step and mainly through the interactive graphic and the dynamic dragging of the point A does the student succeed in finding a new hypothesis and following it (T. 4–8: "But you can also change the *[drags point A vertically*] ehm depth (...) [*moves point A now vertically and horizontally*] (...) So I think that there is also a point where both they do not look the same."). He, therefore, formulates a content-related assumption that mainly constitutes through the use of the interactive graphic (*pivot sign*), and, thus, can be categorized as "content-related reasoning triggered by the use of the structural element " (2a). As a result, he notes – triggered by the interactive graphic – that "you cannot see at first glance that they [two rectangles] are the same size, but that the number of boxes inside is the same" (T. 9–12). Here, the student no longer argues on a level that deals with the activity with the interactive graphic, but on a content-related level (*mathematics signs*) – triggered by the exploratory nature of the task. Thus, the student no longer argues referring to the structural element, but on a mathematical level and, accordingly, his statement be categorized as content-related reasoning (2b).

SUMMARY AND OUTLOOK

Overall, this contribution revealed two effects regarding digital mathematics textbooks and their uses by students: first, the analysis of the digital textbooks pointed out that structural elements different to the structural elements for traditional mathematics textbooks could be identified. On the one hand, the technological realization of exercises resulted in a broader categorization of exercises (*drag and drop* mode, calculation exercises, interactive exercises etc.); on the other hand, and also more importantly, the analysis displayed structural elements (animation, exploration) that do not conform to the structural elements for traditional mathematics textbooks as they originate only due to the digital nature of the textbook and, therefore, had not been realized before in an analogue textbook. As a result, this research project contributes to the diversification of the structural elements for mathematics textbooks. Secondly, on the empirical level, we highlighted different learner usages of the textbook. On the one hand, different structural elements were selected by the student in the context of reasoning indicating that the categories describing student activities with traditional textbooks need to be extended. The second example illustrated the process of reasoning by the user on different usage-related levels during the examination of the structural element highlighting the influence of digital structural elements on the generation of mathematical hypotheses. These first findings must be validated with further empirical data. In this respect, a detailed look at the use of the various options for controlling one's own results (displaying the solution, showing the solution path, dynamic verification of the results) might be interesting.

References

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic meditation in the mathematics classroom. Artifacts and signs after a Vygotskian perspective; in: L. Englisch (Ed.), *Handbook of International Research in Mathematics Education: Second edition* (pp. 746–783). New York: Routledge.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education*, *45*(5), 633–646.

Hornisch, B., et al. (2017). Brockhaus Lehrwerke: Mathematik 5. Klasse. Brockhaus.

Mayring, P. (2008). *Qualitative Inhaltsanalyse*. Weinheim, Basel: Beltz.

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. I. (2016). E-Textbooks in/for teaching and learning mathematics. In L. D. English, & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education: Third Edition* (pp. 636–661). New York: Routledge.

Pohl, M., & Schacht, F. (2018). Digitale Mathematikschulbücher hands-on. Eine Analyse digitaler Mathematikschulbücher und den Schülernutzungen in der Sekundarstufe I exemplarisch an einem Lehrwerk. In M. Schuhen & M. Froitzheim (Hrsg.), *Das Elektronische Schulbuch 2017. Fachdidaktische Anforderungen und Ideen treffen auf Lösungsvorschläge der Informatik*. Münster: LIT Verlag. 119–138.

Rezat, S. (2008). Die Struktur von Mathematikschulbüchern. *Journal für Mathematik-Didaktik*, *29*(1), 46–67.

Rezat, S. (2009). *Das Mathematikbuch als Instrument des Schülers: Eine Studie zur Schulbuchnutzung in den Sekundarstufen.* Wiesbaden: Vieweg+Teubner.

Rezat, S. (2011). Wozu verwenden Schüler ihre Mathematikschulbücher? Ein Vergleich von erwarteter und tatsächlicher Nutzung. *Journal für Mathematik-Didaktik*, *32*(2), 153–177.

INVESTIGATING STUDENTS' ENGAGEMENT DURING MATHEMATICS INSTRUCTION: ANALYZING INTERACTIVE TEXTBOOK WORK WHILE LEARNING FRACTIONS

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Electronic textbooks in mathematics lessons offer new ways to assess and analyze students' engagement in natural school contexts. In this study we validate a combined quantitative (i.e., time on task) and qualitative (i.e., on topic, mathematically correct) analysis of students' electronic textbook-use as a measure for their engagement during mathematics instruction. Cluster analysis based on logfile data of 253 six-graders—who worked on three writing-to-learn activities during fraction instruction—revealed four different Engagement Types. Analyses showed that achievement in a fraction post-test differed between these four groups with more engaged students reaching higher outcomes. Our innovative approach offers a viable way to measure students' engagement during mathematics instruction.

INTRODUCTION

Students' engagement is a pivotal factor in explaining the link between classroom instruction and learning (e.g., Seidel, 2014). Studies regarding learning mathematics have shown that better engagement is associated with higher mathematics achievement (Barkatsas, Kasimatis, & Gialamas, 2009; Singh, Granville, & Dika, 2002). Yet, assessing engagement is challenging, since it requires an observation not only of the final results—e.g., test scores—but should also refer to the process of learning. One viable way to operationalize students' engagement during mathematics instruction is their textbook-use, which has shown to be a reliable predictor for course outcomes (Junco & Clem, 2015), regardless of whether traditional textbooks or electronic textbooks are used (Daniel & Woody, 2013).

Electronic textbooks presented on tablet PCs offer the possibility to observe the learning process of every single child through *logfile data* or *process data*, which is individualized and is gathered throughout all work with the electronic textbooks (Goldhammer, Naumann, Rölke, Stelter, & Tóth, 2017). Firstly, this data allows for assessing students' engagement *quantitatively*, e.g., through the numbers of pages read or the number of bookmarks placed (Junco & Clem, 2015), as well as through measuring the time spent on exercises or instructional content (Hoch, Reinhold, Werner, Richter-Gebert, & Reiss, 2018). In addition, students' input can be rated regarding its *quality*. While recent research focused on quantitative indicators of students' textbook use (Daniel & Woody, 2013; Junco & Clem, 2015), combining both quantitative and qualitative data can offer a broader measure of engagement.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 45–50). Paderborn: Universitätsbibliothek Paderborn.

In mathematics education, one way to measure the quality of students' engagement is the analysis of written statements to questions asked in textbooks: On the one hand, the *written outcome* can be used to assess the knowledge of students. On the other hand, the *writing process* can be analyzed to gain additional indicators for the quality of students' engagement, since this process *itself* can be beneficial for learning. This knowledge-constituting function of writing is referred to as *writing-to-learn* (Keys, 1999). The main idea behind writing-to-learn is that writing can help students to construct, elaborate and structure knowledge, which can be especially beneficial in learning mathematics (Strohmaier, Vogel, & Reiss, 2018). Thus, writing can be considered a learning activity in mathematics education which can be easily implemented in electronic textbooks. This implementation offers the possibility to get insights into students' learning processes, since the writing process itself can logged by the device (Hoch, Reinhold, Werner, Richter-Gebert, et al., 2018). Compared to other forms of learning based on oral communication, reading or mental activities, the data gathered during writing is easily assessible, can be unambiguously assigned to single students and can be analyzed afterwards.

THE PRESENT STUDY

We assume that both the quantity and quality of writing activities are indicators for students' engagement in learning mathematics with electronic textbooks within real classroom situations: successful writing-to-learn can be characterized not only by quantitative measures (e.g., time spent on the activity), but also on the quality of the written text (Strohmaier et al., 2018). Here, we judge text quality in mathematics learning from three basic perspectives: (1) *Context:* Is the written text about mathematics? (2) *Correctness:* Is the written statement mathematically valid? and (3) *Corpus:* Does the written text make use of a mathematical language? Even though the focus on writing-to-learn tasks only sheds light on one particular kind of learning activity, it offers a unique possibility to be observed in learning mathematics with electronic textbooks.

In the present study, we focus on the following explorative research questions to validate a combined quantitative and qualitative analysis of students' textbook-use as a measure for their engagement during mathematics instruction:

1. Which different groups of students can be identified based on their engagement in writing-to-learn activities during mathematics instruction?

2. To what extent can those engagement-types explain mathematical achievement after classroom instruction?

METHOD AND SAMPLE

We used data from our research project *ALICE:fractions* in which we conducted a four-week intervention (15 lessons) on fractions in grade six classrooms in German public schools. Here, we focus on 253 students from the "Tablet PC group" who worked with our interactive environment on iPads (see Hoch, Reinhold, Werner, Reiss,

& Richter-Gebert, 2018, for a detailed view of the project). This digital learning environment was developed as an electronic textbook with interactive content. It allows for hands-on activities and includes adaptive exercises which focus on transitions between various non-symbolic and symbolic representations of fractions and provide intuitive pathways to core fraction concepts.

Here, we focus on students' process data from three writing-to-learn activities during the development of fraction concepts: *Part of the whole* ("How did she divide the pizza?"), *Expanding and reducing* ("How can you easily get fractions with the same value?"), and *Mixed fractions* ("How do you know whether a fraction is more or less than one?"). Students were asked to write down their ideas and to discuss their notes with their neighbor *after finishing* the writing process. They were presented these questions in full screen exercises on a 12.9" iPad during regular classroom work. In these exercises, students could enter their ideas into a text field using the operating system's on-screen keyboard.

DATA

Data were logged whenever students exited an exercise or requested erasure of their input. The interactive exercises recorded the contents of the text fields and the time spans between the time of logging and the last action (logging of data or opening of the exercise) as process data. Process data were saved on the iPad for the duration of the intervention. Afterwards, they were transmitted to a university server and then parsed into a table-like structure for the use with statistics software.

Students' data was coded in five categories: Three qualitative categories *CONTEXT* (answer is on topic; 0 or 1; inter-rater reliability: Cohen's $\kappa = .85$), *CORRECTNESS* (answer is correct, 0 or 1, $\kappa = .80$), *CORPUS* (mathematical language is used; 0 or 1; $\kappa = .75$), as well as two quantitative categories *WORDS* (number of words on topic; counted via spaces; machine coded), and *TIME* (time on task; as measured by the interactive exercise; machine coded). Standardized values of these categories were used for an explorative *k*-means clustering, using Calinski-Harabasz stopping rule to determine the appropriate number of clusters (see Backhaus, Erichson, Plinke, & Weiber, 2018, for a broad overview of common methods for cluster analysis).

To assess students' mathematical achievement, we used a paper-based pre-test covering prior knowledge of fractions (10 items, reliability: McDonald's $\omega = .82$) and a paper-based post-test covering both procedural and conceptual knowledge for basic fraction concepts (21 items, $\omega = .82$).

RESULTS

On average, students worked about 168 seconds on a writing-to-learn activity ($SD = 132$), and wrote about 9.86 words per item ($SD = 9.62$). A total of 75.4% of students' statements were on topic ($SD = 29.3$), 16.1% were mathematically correct

(*SD* = 24.2), and in 16.2% of students' statements mathematical language was used (*SD* = 24.5).

We asked which different groups of students could be identified based on their engagement in writing-to-learn activities during mathematics instruction. Cluster analysis revealed four *Engagement Types* which we describe in comparison to the whole sample (Figure 1): *Industrious Solvers* (*n* = 28; red dots) worked on topic, gave correct answers and used mathematical language. They wrote down the most detailed answers and also spent the most time on the tasks. *Efficient* (*n* = 64; blue squares) also worked on topic, gave correct answers and used mathematical language while they wrote less words and spent less time on the tasks. *Non-Solvers* (*n* = 82; green triangles) still worked on topic, but gave less correct answers using less mathematical language. They also wrote short answers and worked rather briefly on the tasks. *Distracted* (*n* = 79; purple diamonds) did not work on topic, gave mostly wrong answers with very few mathematical vocabulary, only few words and they spent the least time on the tasks.

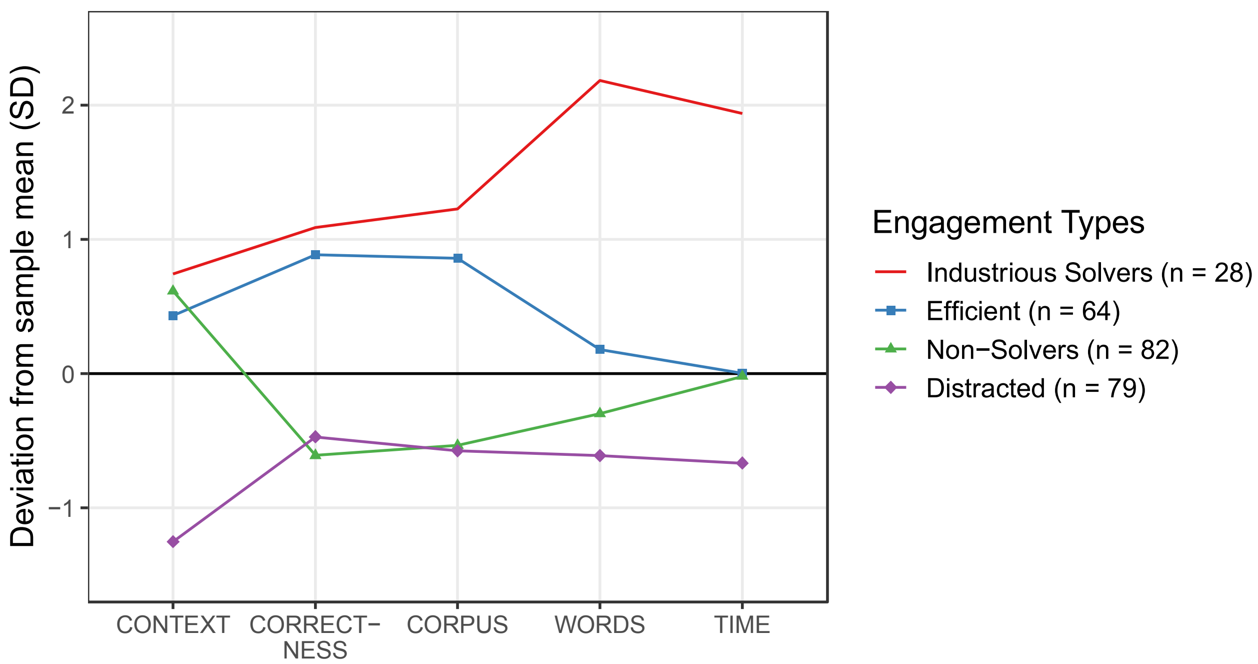

Figure 1. Cluster centers of the four *Engagement Types*, resulting from the cluster analysis of 253 students based on three qualitative measures (CONTEXT, CORRECTNESS, & CORPUS) and two quantitative measures (WORDS, TIME) for students' responses in writing-to-learn activities.

Furthermore, we asked to what extent these Engagement Types can explain mathematical achievement after instruction. If textbook-use is a valid indicator for engagement during classroom instruction, students from different Engagement Types should differ in their outcomes measured after tuition. As can be seen in Table 1, post-test scores differed between the Engagement Types, with the *Industrious Solvers* reaching the highest scores and the *Non-Solvers* reaching the lowest scores.

		Pre-Test		Post-Test	
Engagement Type	*N*	*M*	*SD*	*M*	*SD*
Industrious Solvers	28	.48	.28	.74	.18
Efficient	64	.47	.29	.61	.21
Non-Solvers	82	.33	.28	.40	.25
Distracted	79	.40	.29	.46	.26

Table 1. Achievement in pre-test and post-test for each Engagement Type.

There is a significant large effect of Engagement Type on the post-test score after controlling for the pre-test score, $F(3, 248) = 18.4$, $p < .001$, $\eta^2 = .182$. Post-hoc Tukey contrasts showed that all but two Engagement Types differed pairwise significantly, ps < .05, while no significant difference between the *Non-Solvers* and the *Distracted* was found, $p = .849$.

DISCUSSION

Our results support the assumption that quantitative and qualitative process data reflecting students' mathematic textbook-use in real classroom scenarios can be used as an indicator for students' engagement during mathematics instruction. We could identify four different Engagement Types based on five theoretically described categories: *Industrious Solvers* (highest engagement), *Efficient*, *Non-Solvers*, and *Distracted* (lowest engagement). We conclude that students in these clusters show different levels of engagement in a descending order. This conclusion is supported by the results of the achievement test conducted after working with the electronic textbook. Here, students from higher engaged clusters reach better outcomes than students from lower engaged clusters.

Our results are in line with other studies regarding quantitative measures of students' textbook-use and course outcomes in mathematics (e.g., Daniel & Woody, 2013; Junco & Clem, 2015): Even though the *Industrious Solvers* and the *Efficient* showed comparable prior knowledge and differed only in the number of words written and the time spent on task, the *Industrious Solvers* outperformed the *Efficient* in the post-test.

In contrast to past studies, we also used qualitative indicators for students' engagement, regarding the context, the correctness and the corpus of their answers to specific mathematical writing-to-learn activities. This approach made distinctions between students showing comparable quantitative engagement possible: Despite a comparable number of words written during the exercises and a comparable time spent on task, the *Efficient* outperformed the *Non-Solvers* in the post-test.

We conclude that our approach—i.e., using interactive textbooks as a measurement device and considering both quantitative and qualitative process data gathered during electronic textbook-use—offers a new and viable way of assessing students' engagement in mathematics education. This offers rich possibilities to measure students' engagement in natural school contexts, its relation to learning processes, and its role in digitally-supported learning environments. It therefore can support both research and practice in mathematics education.

References

Backhaus, K., Erichson, B., Plinke, W., & Weiber, R. (2018). *Multivariate Analysemethoden* [Multivariate Data Analysis] (15th ed.). Berlin, Heidelberg: Springer.

Barkatsas, A. (Tasos), Kasimatis, K., & Gialamas, V. (2009). Learning secondary mathematics with technology: Exploring the complex interrelationship between students' attitudes, engagement, gender and achievement. *Computers & Education*, *52*(3), 562–570. doi:10.1016/j.compedu.2008.11.001

Daniel, D. B., & Woody, W. D. (2013). E-textbooks at what cost? Performance and use of electronic v. print texts. *Computers & Education*, *62*, 18–23. doi:10.1016/j.compedu.2012.10.016

Goldhammer, F., Naumann, J., Rölke, H., Stelter, A., & Tóth, K. (2017). Relating product data to process data from computer-based competency assessment. In D. Leutner, J. Fleischer, J. Grünkorn, & E. Klieme (Eds.), *Competence assessment in education* (pp. 407–425). Cham: Springer.

Hoch, S., Reinhold, F., Werner, B., Reiss, K., & Richter-Gebert, J. (2018). *Fractions. Getting in touch with rational numbers* (Apple iBooks Version). Munich, Germany: Technical University of Munich. Retrieved from http://go.tum.de/623496

Hoch, S., Reinhold, F., Werner, B., Richter-Gebert, J., & Reiss, K. (2018). Design and research potential of interactive textbooks: the case of fractions. *ZDM Mathematics Education*, *50*(5), 839–848. doi:10.1007/s11858-018-0971-z

Junco, R., & Clem, C. (2015). Predicting course outcomes with digital textbook usage data. *The Internet and Higher Education*, *27*, 54–63. doi:10.1016/j.iheduc.2015.06.001

Seidel, T. (2014). Angebots-Nutzungs-Modelle in der Unterrichtspsychologie. Integration von Struktur- und Prozessparadigma [Utilization-of-learning-opportunities models in the psychology of Instruction: integration of the paradigms of structure and of process]. *Zeitschrift für Pädagogik*, *60*(6), 850–866.

Singh, K., Granville, M., & Dika, S. (2002). Mathematics and Science Achievement: Effects of Motivation, Interest, and Academic Engagement. *The Journal of Educational Research*, *95*(6), 323–332. doi:10.1080/00220670209596607

Strohmaier, A., Vogel, F., & Reiss, K. M. (2018). Collaborative epistemic writing and writing-to-learn in mathematics: can it foster mathematical argumentation competence? *Ristal*, (1), 135–149. doi:10.23770/rt1817

ANALYSING THE EFFECTIVENESS OF A COMBINATION OF DIFFERENT TYPES OF FEEDBACK IN A DIGITAL TEXTBOOK FOR PRIMARY LEVEL

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Feedback is widely acknowledged as an important influential factor on learning and achievement. The fact that interactive digital learning tools constantly provide feedback to learners' actions with the contents is indeed one of the most emphasized advantages of learning with digital tools. While there is a growing body of research related to adaptive feedback based on artificial intelligence in education, there still is the need to understand less sophisticated ways of implementing feedback into educational technologies like digital textbooks, which promise to be implementable at a large scale. The study presented in this paper, analyses the effectiveness of a particular combination of different feedback types that are offered in a digital mathematics textbook for the elementary level. The results show a low effectiveness of the different types of feedback. Based on this result, possibilities of developing and evaluating more effective feedback in digital textbooks are outlined.

INTRODUCTION

Feedback is widely acknowledged as an important influential factor on learning and achievement (Hattie & Timperley, 2007). According to Hattie and Timperley (2007, p. 81) feedback is understood as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding". The goal of feedback is to support understanding and/or performance. In line with this, Shute (2008, p. 154) defines formative feedback as "information communicated to the learner that is intended to modify his or her thinking or behavior for the purpose of improving learning". According to Hattie and Timperley (2007, p. 87) effective feedback has to address three questions: "Where am I going? How am I going? Where to next?".

The fact that interactive digital learning tools constantly provide feedback to learners' actions with the contents is indeed one of the most emphasized advantages of learning with digital tools (e.g. Mason & Bruning, 2001). In fact, by definition interactivity means that users get immediate feedback to their actions with the tool.

Research related to feedback aims at identifying features of feedback that increase its efficiency. Two aspects seem to be important for effective feedback: 1) the forms of feedback and 2) the timing of feedback. Research has shown that both, the wrong form of feedback and the wrong timing might even have negative effects on learning and achievement (Fyfe & Rittle-Johnson, 2016a; Hattie & Timperley, 2007). The majority of studies in this context quantitatively measure and compare effect sizes of

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different forms or timings of feedback in order to draw conclusions regarding its effectiveness. The underlying assumption in these settings is that students react consistently to the respective form or timing of feedback. Another research area has been to identify further influences on the effectiveness of feedback, e.g. prior knowledge (Fyfe & Rittle-Johnson, 2016b) or feedback specificity (c.f. Shute, 2008). This research shows that individual factors influence the effectiveness of feedback. Acknowledging that learning processes are individual and might differ among learners leads to the question if different learners need different kinds of feedback in order to receive support for their learning processes. A research focus in this context has been adapting digital learning environments to learner characteristics by artificial intelligence in terms of adaptive learning trajectories, adaptive feedback, intelligent tutors, and pedagogical agents. While there is a growing body of research related to artificial intelligence in education (Grandbastien, Luckin, Mizoguchi, & Aleven, 2016), the majority of digital learning tools and textbooks offered on the market and on the web differs substantially from the state of the art of research in this area. There still is the need to understand less sophisticated ways of implementing feedback into educational technologies like digital textbooks, which promise to be implementable at a large scale. In the context, the effects of combining different kinds of feedback in a digital tool and thus confronting learners with multiple kinds of feedback while working on tasks from the digital textbook have rarely been investigated.

The aim of this paper is to analyse the effectiveness of a particular combination of different feedback types that are offered in a digital mathematics textbook for the elementary level published by a German textbook publisher. The research question is: What is the effect of the different types of feedback offered in a digital textbook on learners' behaviour?

THEORETICAL FRAMEWORK

Hattie and Timperley (2007) provide a model of feedback in which they differentiate between four levels of feedback: the task level, the process level, the self-regulation level, and the self level. Feedback at the task level informs about how well the task is understood or performed. At the process level, feedback refers to the main processes needed in order to understand or perform the task. Feedback at the self-regulation level relates to self-monitoring and the direction and regulations of actions, whereas feedback at the self level conveys personal evaluations and affect about the learner. The study presented in this paper refers to feedback on the task level.

In order to differentiate different forms of feedback, which are offered by a digital tool, the study presented in this paper refers to a classification of feedback according to Shute (2008). She distinguishes different types of feedback according to their complexity. For the study presented in this paper, the following types are relevant:

1. Knowledge of results feedback (KR) informs the learner about the correctness of an answer;

2. Knowledge of correct response (KCR) feedback informs the learner about the correct response;
3. Repeat-until-correct (RUC) feedback informs the learner about an incorrect response and offers the possibility of new try to answer the task;
4. Location of mistakes (LOM) feedback informs the learner about the location of an error in the solution without giving the correct response;
5. Elaborated feedback (EF), which offers further information regarding the solution of the task or the solution of the learner. For the study presented in this paper hints/cues/prompts are the relevant type of EF, which contain information guiding the learner in the right direction.

METHODOLOGY

The textbook

In the study, the textbook "Denken und Rechnen interaktiv" was used. Currently, it is one of the very few mathematics textbooks for the elementary grades on the German market that offers feedback and interactive answer formats like multiple choice or drag and drop. It is developed by a German publisher for textbooks, which also offers a printed version of the book. The structure and the contents of the printed and the digital version are analogous.

The digital version of the textbooks offers different kinds of feedback:

1. After entering an answer, KR-feedback is presented combined with RUC-feedback. Regularly, the student has three tries for each task. If there is only one answer to enter the textbooks shows a green field with a positive or a red field with a negative feedback message. If there are multiple entries to make, the textbook provides error flagging feedback in that it highlights the corrects answers in green and the wrong answers in red. The wrong answers disappear before the next try, the corrects answers stay.
2. If the student has not succeeded after the second try, a symbol with a lightbulb appears on the screen. A click on the symbol provides a standard EF message for students, which contains a hint, cue or prompt for solving the task.
3. If the student has not succeeded to answer the task correctly after the third try, KCR-feedback is provided.
4. After solving a set of tasks, a summative feedback is provided, which provides information about the time needed for working on the task and a summary of the number of tasks that were solved after the first, the second, and the third try, respectively, or not at all. This type of feedback is not included in the study, since it is not on the task level.
5. The textbook offers a lexicon of mathematical terms and procedures, which is accessible all throughout the whole process of working on a task.

Mathematical terms in the wording of the task are directly linked with the lexicon. This is also regarded as EF.

Data collection and analysis

Data was collected by videotaping third grade students from different German primary schools working with the digital textbook. Students were working individually with the digital textbook either in experimental settings or in a whole class setting. Data was analysed using MAXQDA by coding the success of the following try after the students got feedback from the textbook or used a scaffold. In order to analyse the effect of the feedback on students' behaviour, only students, which answered the task incorrectly in the first try were included in the analysis. Altogether, the data of 27 students with 362 tries was analysed. This procedure was chosen, because the log-files from the digital textbook were not accessible.

RESULTS

Table 1 shows the number of correct and incorrect tries that followed different kinds of feedback.

Feedback	Following try				
	Correct	Incorrect	∑	Correct (%)	Incorrect (%)
KCR	10	26	36	28	72
KR	49	87	136	36	64
EF	13	43	56	23	77
Sum	**72**	**156**	**228**	**32**	**68**

Table 1. Number of correct and incorrect tries following feedback

Table 1 shows that on the average the different kinds of feedback only yielded a correct result in approx. one third of the cases. After KCR and EF it was even only approx. one fourth of the following tries that were solved correctly. Since EF is presented after the second unsuccessful try, students who received EF- or KCR-feedback had already received KR-feedback before. Therefore, the results for KCR and EF in fact relate to the combinations of KR with EF and KCR. In sum, these results question the effectiveness of the feedback combination in the digital textbook and give rise to the question of reasons for these results.

DISCUSSION

Previous research on the effectiveness of feedback has shown that KR- and KCR-feedback are not very effective and that EF is more effective than the other two types. However, the combination of different kinds of feedback in one digital tool has rarely been a matter of study. The results of this exploratory study give rise to the hypothesis that the combination of different kinds of feedback as implemented in the digital textbook “Denken und Rechnen interaktiv” does not increase the effectiveness. Nevertheless, the fact that digital textbooks and tools usually combine different kinds of feedback point to a research potential related to digital textbooks and tools. The task of this research could be developing a combination of different kinds of feedback that is effective for learners when working with the digital textbook. Developing a better understanding of the processes when students are working with digital textbooks might yield to feedback that better adjusts to problems students encounter within these processes.

Furthermore, the low effectiveness of EF in the digital textbook in the present study gives rise to a second research potential related to digital textbooks. In a previous exploratory qualitative study (Rezat, 2017) it was shown that the feedback message is subject to interpretation by the students and requires the activation of relevant concepts in order to be effective. Because of the low effectiveness of the EF in the digital textbook, it is likely that the information presented in EF does not meet the problems of the learners with the tasks. Therefore, it is important first, to analyse the problems that students show with particular tasks, second, developing feedback that relates to the problems identified in the first step, and third, evaluating the effectiveness of the feedback that is presented to students.

In both cases, the possibility to control the conditions of the provision of feedback and to trace students’ behaviour following the feedback in the log-data of digital tools point to the research potential that is related to digital textbooks (Hoch, Reinhold, Werner, Richter-Gebert, & Reiss, 2018).

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References

Fyfe, E. R., & Rittle-Johnson, B. (2016a). The benefits of computer-generated feedback for mathematics problem solving. *Journal of Experimental Child Psychology, 147*, 140–151. http://dx.doi.org/10.1016/j.jecp.2016.03.009

Fyfe, E. R., & Rittle-Johnson, B. (2016b). Feedback both helps and hinders learning: The causal role of prior knowledge. *Journal of Educational Psychology, 108*(1), 82–97. Retrieved from http://psycnet.apa.org/index.cfm?fa=buy.optionToBuy&uid=2015-25422-001

Grandbastien, M., Luckin, R., Mizoguchi, R., & Aleven, V. (2016). Preface to the IJAIED 25th Anniversary Issue. *International Journal of Artificial Intelligence in Education, 26*(1), 1–3. https://doi.org/10.1007/s40593-015-0092-6

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77*(1), 81–112. https://doi.org/10.3102/003465430298487

Hoch, S., Reinhold, F., Werner, B., Richter-Gebert, J., & Reiss, K. (2018). Design and research potential of interactive textbooks: the case of fractions. *ZDM Mathematics Education, 50*(5), 839–848. doi:10.1007/s11858-018-0971-z

Mason, B. J., & Bruning, R. H. (2001). Providing feedback in computer-based instruction: what the research tells us. In *Class research report no. 9.* Lincoln: Center for Instrnctional Innovation, University of Nebraska-Lincoln.

Rezat, S. (2017). Students' utilizations of feedback provided by an interactive mathematics e-textbook for primary level. In T. r. s. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1–5, 2017)* (pp. 3724–3731). Dublin, Ireland: Dublin City University, Institute of Education, and ERME.

Shute, V. J. (2008). Focus on Formative Feedback. *Review of Educational Research, 78*(1), 153–189. doi:10.3102/0034654307313795

Symposium B: Cross-cultural research on teachers' use of resources

CROSS-CULTURAL RESEARCH ON TEACHERS' USE OF RESOURCES

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PURPOSE

This symposium brings together researchers who work across cultural contexts to understand teachers' use of resources in mathematics teaching. The goal of the symposium is to illustrate and learn from different approaches to conceptualizing and examining this phenomenon and the role that cultural traditions and practices play in both the work of resource use and research on it. For this reason, contributions to the symposium examine more than one cultural context. The focus of the symposium is relevant, given contemporary tendencies toward globalization in educational practice and policy and increased interest in learning from research across cultural boundaries.

The following questions guide the symposium:

1. What can cross-cultural studies on resource use offer the field?
2. How are different approaches to conceptualizing and examining teachers' use of resources to teach mathematics illuminated through cross-cultural analysis?
3. What methodological challenges emerge when undertaking cross-cultural research on teachers' resource use?

ORGANIZATION

The symposium consists of four invited contributions (see below) and is structured in the following way: 10 minutes introduction, 15 minutes presentation and 5 minutes discussion for each contribution, 25 minutes broader discussion addressing the three research questions, and 5 minutes closing down.

INVITED CONTRIBUTIONS

The following invited contributions are part of the symposium:

Takeshi Miyakawa & Stéphane Clivaz

Japanese and Swiss pre-service teachers' resources in the cross-cultural collaborative design of a mathematics lesson

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Tuula Koljonen

Finnish and Swedish elementary school teachers' interplay with Finnish curriculum resources: an attempt at unraveling tacit cultural practices

Janine Remillard, Hendrik Van Steenbrugge, Heidi Krzywacki, Kirsti Hemmi, Rowan Machalow, Tuula Koljonen, & Yanning Yu

A cross-cultural study on teachers' use of digital resources in Sweden, Finland, the USA, and Flanders

Jana Višňovská

Notes from designing one resource for teachers' use across the contexts of Mexico, Australia, and South Africa

FINNISH AND SWEDISH ELEMENTARY SCHOOL TEACHERS' INTERPLAY WITH FINNISH CURRICULUM RESOURCES: AN ATTEMPT AT UNRAVELING TACIT CULTURAL PRACTICES

Tuula Koljonen

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This study investigates tacit cultural practices in Finland and Sweden by studying the lesson structure and questioning strategies teachers use in mathematics lessons. The data consists of four teacher cases of both region, each based on three video-recorded mathematics lessons and a complementary interview with each teacher. The teachers in both countries use an originated Finnish curriculum program which have a potential to influence the lesson realization. The initial analysis reveals that although teachers in both countries were using Finnish programs, the Swedish lessons display versions of individualized learning pedagogy while the Finnish lessons display a form of differentiated teaching pedagogy. The results add to knowledge from two different educational contexts about tacit cultural practices that may otherwise be unnoticed.

INTRODUCTION

Research raises the importance of curriculum resources, such as curriculum programs (commercially produced textbooks and teacher guides), as they largely shape what, and how, mathematics is to be taught and learned (e.g. Fan, Zhu, & Miao, 2013) in ways that reflect cultural values (e.g. Haggarty & Pepin, 2002). In our previous studies, we found a pattern of altogether six recurrent activities within each lesson: (1) teacher-led classroom instruction; (2) mental calculation; (3) problem-solving; (4) games and playing activities; (5) homework; and (6) students working in the textbook when we examined the structure and content of Finnish teacher guides and, the idea in the guides are that the recurring lesson elements comprise a collection of ideas that teachers can choose to use or not in the classes and, no lesson script is offered that a teacher could follow as such (Hemmi, Krzywacki, & Koljonen, 2017). We have further, examined what kind of constructed mathematics classroom practice they may potentially construe and we identified three norms embedded in them: (1) creating opportunities for learning through a variety of activities and communication; (2) keeping the class gathered around a specific mathematical topic; and (3) concurrent active involvement of teachers and students (Koljonen, Ryve, & Hemmi, 2018). However, these potential classroom characteristics suggested by the teacher guides were not visible in the case study of one Swedish teacher's interaction with a Finnish teacher guide (Koljonen, 2017). Finnish curriculum programs are being used in other countries such as Sweden and Italy. Few, if any studies have investigated how curriculum programs created in one country are used by teachers in another country and how that interaction influence classroom practice. I anticipate that a study on the use of curriculum programs from one culture by teachers from another culture will advance our understanding of the

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cultural practices in both cultures, and of the interplay between teacher and curriculum programs. Finland and Sweden provide valuable cases because they have many similarities, including geographic proximity, an inclusive free and compulsory basic education starting at age seven, an absence of tracking students by performance level, and teachers in both countries choose freely what program and which components within and outside the program to use.

It is impossible to find two teachers who teach in exactly the same way. Yet, in a given school, community, or country, there are patterns in the social activity of teaching that are characteristic for them. The theoretical approach in this study rests on cultural norms. Cultural norms are the regularities of the practice and the social interaction established by a group, regarding what is acceptable or desirable. The values of that cultural practice shape norms, and involve a taken-as-shared idea of what constitutes an appropriate and desirable mathematics classroom (cf. Hiebert et al., 2003). Teachers are a part of those embedded social practices together with the curriculum programs which are presumed to legitimise and reflect the different cultural specific educational values of countries (cf. Haggarty & Pepin, 2002) and thus reflect the specific character of the teaching and learning activities potentially realised in classrooms. Curriculum programs serve as an important tool for teachers in both enabling and constraining their thoughts and actions (Stein, Remillard, & Smith, 2007) in implementing ideas about teaching. Both how teachers structure their lessons and how they enable student's participation through usage of questioning strategies reflects the specific educational and cultural norms valued within a country. Several researchers claim e.g., that providing opportunities for participation in subject-related situations is among the most important aspects of teaching and teachers' elaboration of students' ideas play an important role e.g. in high-quality teaching (Klette et al., 2018). To deepen our understanding of how cultural norms may or may not influence teacher-curriculum interplay and the enacted classroom, it is of interest to investigate Finnish and Swedish teachers' practices when using a same kind of (i.e., a Finnish) program while teaching. The research question guiding this study is: *what tacit cultural practices do Finnish and Swedish mathematics lessons display when teachers are using an originally Finnish teacher's guide?*

METHODOLOGY

The research design entails a multiple case study (Yin, 2009) of 4 Finnish and 4 Swedish teachers where each teacher forms a case. The data consist of 24 videotaped mathematics lessons: three consecutive mathematics lessons per teacher (40-60 minutes/lesson) and 1 audio-recorded semi-structured teacher interview (50-110 minutes). All the 8 participating primary school teachers: 1) held formal education as mathematics teachers, 2) were regarded as locally competent (i.e., nominated by the school principal and the municipality as local subject specialists), 3) worked at practice schools (schools collaboration with universities), and 4) used the same originally Finnish teaching materials. The analysis focuses on the lesson structure and students' participation through interaction opportunities during teachers questioning strategies during whole-class teaching episodes.

I approached the data first by investigating mathematics classrooms of the transcribed videotaped lessons by identifying the structure of the lessons. Two common lesson organisation formats in classrooms around the world (O'Keefe, Xu, & Clark, 2006) were used in this analysis: 1) classwork (CW), the time when teachers are working with all students and usually orchestrating discussions and 2) seatwork (SW), the time when students' work individually or in small groups on assigned tasks and where the talk is mostly private. This provided me a first framing, within which teachers were engaging students in two very different classroom practices. To understand more of what happens in those videotaped classrooms, I needed to go beyond structural patterns of lessons and look at the features of classroom talk in which the teachers implicitly realize their roll in not just teaching of mathematics but also in engaging students to learn. I therefore, focused on the interaction patterns through teachers' questioning strategies during the periods of classwork. For that analysis I used Boaler and Brodies (2004) Teacher Questions Coding Scheme as it provides a compilation and descriptions of various question types that teachers could ask students. This helped me to identify the use of lesson time for the different types of activities and interactions in the classroom environment. In this study, the interviews were used as supplementary data for triangulation of the analysis of the classroom practices by confirming or not confirming what were displayed on the videotaped lessons.

RESULTS

When categorizing the videotaped mathematics lessons as either classwork (CW) or seatwork (SW), there were substantial difference between how the Finnish and Swedish lessons were organized. **CW** was the dominating style (10/12) in the Finnish videotaped lessons while only one of the lessons involved mostly *SW* (see Table 1). Whereas *SW* (9/12) was dominant in the Swedish videotaped lessons while two of the lessons were mostly spent as **CW** (see Table 2).

	Teacher grade1			Teacher grade 3			Teacher grad 4			Teacher grad 5		
	L1	**L2**	**L3**	**L1**	**L2**	**L3**	**L1**	**L2**	**L3**	*L1*	**L2**	L3
CW	100	85	60	100	67	69	74	65	58	29	62	51
SW	0	15	40	0	33	31	26	35	42	71	38	49

Table 1. Percentage of classwork and seatwork of the Finnish teachers' lessons

	Teacher grade 1			Teacher grade 3			Teacher grade 4			Teacher grade 5		
	L1	**L2**	*L3*	*L1*	*L2*	**L3**	L1	*L2*	*L3*	*L1*	*L2*	*L3*
CW	32	71	29	28	31	78	46	32	38	35	28	37
SW	68	29	71	72	69	22	54	68	62	65	72	63

Table 2. Percentage of classwork and seatwork of the Swedish teachers' lessons

All the Finnish lessons were 45 minutes in length. The analyzed Finnish lessons did not portray one typical coherent lesson type. Instead, teachers and students were simultaneously involved in several loops of instruction and practice sessions during each lesson, which may explain the larger proportion of CW (see table 1). The teacher guides' six recurrent lesson activities (teacher-led classroom instruction; mental calculation; problem-solving; games and play activities; homework; and students working in the textbook) were visible during these sessions and the lessons often ended with a game or other whole-class activity. All students were working with the same mathematical topic and all the Finnish teachers used a document camera during whole CW sessions, so that the students could participate and be at the front when solving tasks and problems during these sessions. The Swedish lessons varied in lesson length (40-60 minutes). All the videotaped Swedish lessons were divided into two distinctive episodes: 1) a teaching session where the teacher leads a *Genomgång* (instruction: going through something) and 2) a practicing session where students work in their workbook. All the Swedish teachers had students working with other curriculum programs and within other mathematical areas than the focus of the observed lessons. They further used SmartBoards or projectors in combination with whiteboards when conducting the *genomgång*, and for that they used the ready-made material (teacher-led classroom instruction) from the guides. The length of the Swedish SW episodes was much longer than the CW length (see table 2) and the Swedish teachers always ended their lessons with SW.

Teachers		Questioning types								
		1.	2.	3.	4.	5.	6.	7.	8.	9.
FIN	T1	xx	x	x	x	xx	x	x	x	xx
	T3	xx	xx	xx	x	x	xx	-	-	-
	T4	xx	xx	-	xx	xx	-	-	-	-
	T5	xx	xx	xx	xx	x	xx	xx	xx	x
SWE	T1	xx	-	-	xx	-	-	-	1	-
	T3	xx	1	-	1	-	-	-	-	-
	T4	xx	1	x	xx	-	-	-	-	-
	T5	xx	x	-	x	x	-	1	-	-

Table 3: Teacher Questions Coding Scheme from Boaler and Brodie (2004): Occurrence of the question types to categories 1-9. xx occurred regularly, x occurred sporadically, - were absent and, 1occured once in three lessons.

When analysing how teachers enable students' participation and interaction during the whole-class teaching episodes, substantial similarities as well as differences among the Finnish and Swedish teachers' usage of questions emerged. I categorized these questions according to Boaler and Brodie's (2004) nine types. Both the Finnish and the Swedish teacher use question type 1 (Gathering information, leading students through a method) and 4 (Probing, getting students to explain their thinking) largely (see table 3). However, Finnish teachers also use question type 2 (Inserting terminology), 3

(Exploring mathematical meanings and/or relationships), 5 (Generating discussion) and 6 (Linking and applying), which were almost absent among the Swedish teachers. Two of the Finnish teachers used all the nine question types including 7 (Extending thinking), 8 (Orienting and focusing), and 9 (Establishing context) during the videotaped lessons. None of the Swedish teachers used all of the nine question types.

DISCUSSION AND CONCLUDING THOUGHTS

This investigation aimed to reveal tacit cultural practices of Swedish and Finnish mathematics classrooms. Teacher guides are, in this study regarded as artefacts used by teachers when designing and enacting teaching, as well as reflecting native cultural values (cf. Haggary & Pepin, 2002). The originally Finnish programs, in this study, are used by teachers both in the original, Finnish context and, in a new, Swedish educational context.

The analysis reveal that the Finnish lessons display a form of differentiated teaching pedagogy in which teachers adjust the learning needs for a group of students by having student concurrently participate during teacher-led activities. The classrooms contain a substantial variation of activities from the guides, which enables students to participate as contributing actors (cf. Klette et al., 2018), supported by teachers' usage and variation of different question types (cf. Boaler & Brodie, 2004). This is in contrast with the Swedish lessons, displaying versions of the individualized learning pedagogy, in which the teachers accommodate learning needs to individual students by having a short introduction, students then work at their own pace in their workbooks and where the participation in the whole class activity, *genomgången* relate to low inference interaction since minimal variations of other activities or teacher question strategies are offered in the Swedish lessons.

Despite the usage of the same curriculum program it is difficult to see similar elements recurring in both the Finnish and Swedish mathematics classrooms and thus, curriculum program influence on the new cultural practice. This may not be of a surprise, since the Finnish teachers are using a curriculum program developed within the Finnish cultural norms and in line with the teaching traditions and social practices norms. Whereas the Swedish teachers use a Finnish program which is developed in another culture. Even so, based on these findings, I conclude that there are cultural similarities between teachers as well as differences within teacher's classroom practices in relation to countries (cf. Haggarty & Pepin, 2002; Hiebert et al., 2003; O'Keefe et al., 2006). I further conclude, that the use of the originally Finnish curriculum program in a new context has not had the intended impact on the practices as promoted by the embedded cultural values of Finnish teacher guides. This challenges the idea of reforming or changing mathematics teaching via launching new curriculum programs. Yet without targeted support for how to use a new material it is hard, even if a teacher is regarded as competent, to independently conduct changed or improved teaching and simultaneously maintain or gain pedagogical autonomy. Further investigations are needed to better understand, more in depth how cultural norms influence teacher-curriculum interplay and the enacted classroom. I will thus investigate how teachers organize the activities from the teachers' guides within each lesson and what kind of teaching strategies teachers apply.

References

Boaler, J., & Brodie, K. (2004). The importance of depth and breadth in the analyses of teaching: A framework for analyzing teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 773–782). Toronto, Canada.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education, 45*(5), 1–14.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks in England, France and Germany: Some challenges for England. *Research in Mathematics Education, (4)*1, 127–144. doi: 10.1080/14794800008520106

Hemmi, K., Krzywacki, H., & Koljonen, T. (2017). Investigating Finnish teacher guides as a resource for mathematics teaching. *Scandinavian Journal of Educational Research*.

Hiebert, J., Gallimore, R., Gamier, H., Bogard Giwin, K., Hollingsworth, H., Jacobs, J. et al. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study.* Washington, DC: National Center for Educational Statistics.

Klette. K., Sahlström, F., Blikstad- Balas, M., Luoto, J. M., Tanner, M., Tengeberg, M. and Roe, A. (2018) Justice through Participation: Student Engagement and in Nordic Classrooms. *Education Inquiry*.

Koljonen, T. (2017). Finnish teaching materials in the hands of a Swedish teacher: The telling case of Cecilia. In T. Dooley & G. Gueudet (Eds,). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1626–1633). Dublin, Ireland: DCU Institute of Education & ERME.

Koljonen, T., Ryve, A., & Hemmi, K. (2018). Analysing the nature of potentially constructed mathematics classrooms through teacher guides – the case of Finland. *Research in Mathematics Education, 20*(3), 295–311.

O'Keefe, C., Xu, L. H., & Clarke, D. (2006). Kikan-shido: Between desks instruction. In D. Clarke, E. Jablonka, J. Emanuelsson, & I. A. C. Mok, *Making connections: Comparing mathematics classrooms around the world* (pp. 73–106). Rotterdam: Sense.

Stein, M. K., Remillard, J. T., & Smith, M. S. (2007). How curriculum influence student learning. In F. K. Lester Jr. (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 319–369). Charlotte, NC: Information Age.

Yin, R. K. (2009). *Kvalitativ forskning, från start till mål*. Lund: Studentlitteratur.

PRE-SERVICE TEACHERS' RESOURCES IN THE CROSS-CULTURAL COLLABORATIVE DESIGN OF A MATHEMATICS LESSON

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This paper investigates the resources the student-teachers used to design and implement grade 4 mathematics lessons in the context of a project-based international exchange programme between Switzerland and Japan. The lesson, initially planned together by nine student-teachers of the two countries, was implemented separately in each country. In spite of the collaborative design of a lesson, its enactment was quite different. The analysis of the resources (namely lesson plan, curricula, and textbooks) allows us to identify the elements that yield the differences between the two lessons.

INTRODUCTION

Since a decade, researchers try to understand mathematics teachers' work and their evolution through the analysis of resources the teachers use or develop for their teaching practices. This approach, called *documentational approach to didactics* (Gueudet & Trouche, 2009), focuses on the use of resources. Resources and their use, or the *document* should be very different according to the educational contexts or the institutions because they are culturally rooted. As teaching practices differ across contexts, the roles of resources, such as a textbook, are also different. We consider that some resources for teachers constitute a *didactic infrastructure* (Chevallard, 2009) that strongly supports and shapes the lesson design.

In this paper, we carry out a comparative analysis of the resources or of the didactic infrastructure for the student-teachers used in the design and implementation of mathematics lessons in two countries, Switzerland and Japan, and investigate related cultural factors that yield the differences of teaching practices identified in our previous comparative study of lessons. The resources we will analyse are, in particular, the lesson plans developed during a project-based student and teacher exchange programme (called PEERS project), as well as the mathematics textbooks and the national or regional curricula which are principal references for the student-teachers.

In what follows, we first present, as a context, the PEERS project and the differences we have identified in a comparative analysis of lessons, and then carry out an analysis of resources, in order to identify the cultural elements that yield such differences.

COLLABORATIVE DESIGN OF A LESSON

We briefly describe our project of the collaborative design of a mathematical lesson, which allows us to mention the collected data, and the results of comparative analysis in our previous study (Clivaz & Miyakawa, to appear).

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 67–72). Paderborn: Universitätsbibliothek Paderborn.

PEERS Project

PEERS project is a student and professor exchange program carried out by Lausanne University of Teacher Education (HEP Vaud). It consists of student exchanges around a jointly defined research project by a group of students from HEP Vaud in association with a group of students from the partner university. Each PEERS is supervised by a teacher-researcher of each institution, combining face-to-face (one week in fall and another week in spring) with distance online collaborative work phases. PEERS with Joetsu University of Education (the previous affiliation of the first author) was supervised by the two authors of this paper.

In our PEERS, the group of students and professors first met through Skype meetings in fall 2017, and decided on the general theme of PEERS and the mathematical theme: the collaborative development of a problem solving geometry lesson for grade 4 pupils following a lesson study process (Hart, Alston, & Murata, 2011). The group spent one week in Joetsu in October 2017 to design a task, to study the topic and to plan the lesson together. At the end of the week, a first draft of the lesson plan was ready. During the winter, the two groups developed their lessons separately and taught them several times in each country. The Japanese group spent one week in Lausanne in February 2018. During this week, the group observed the last Swiss lesson, watched the video of the last Japanese lesson, and discussed the differences and commonalities.

The problem the group selected for designing a lesson was the one in the Swiss textbook (Danalet, Dumas, Studer, & Villars-Kneubühler, 1999). The question is: "Divide a square into several squares, but not more than 20. Find as many solutions as possible". The lesson plan by the Swiss students is available on the websites of Lausanne Laboratory Lesson Study (www.hepl.ch/3LS).

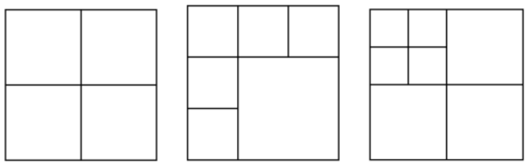

Figure 1. Some of the possible solutions for 4, 6 and 7

Differences between two "same" lessons

In our previous study, we carried out a comparative analysis of the lessons implemented in each country. Even though the task was initially designed collaboratively in the face-to-face workshops in Japan, its implementations in Switzerland and Japan were very different. We found explicit differences, in particular, in the structures of lesson described with the phases (see Figure 2) and in the teacher's way of validating pupil's answers found in each of the corresponding phases.

Concerning the structure, one big difference is that the sharing phase in the Japanese lesson, which is often called *neriage*, is longer than in the Swiss lesson, and includes not only the collective work that the teacher manages in the whole classroom setting, but also the group work, that is to say, the sharing or *neriage* in the Japanese lesson aims not only to share pupil's answers in the classroom but also to develop mathematical ideas in the collective setting. Another difference is that there was no phase for synthesis (or *matome*) in the Swiss lesson.

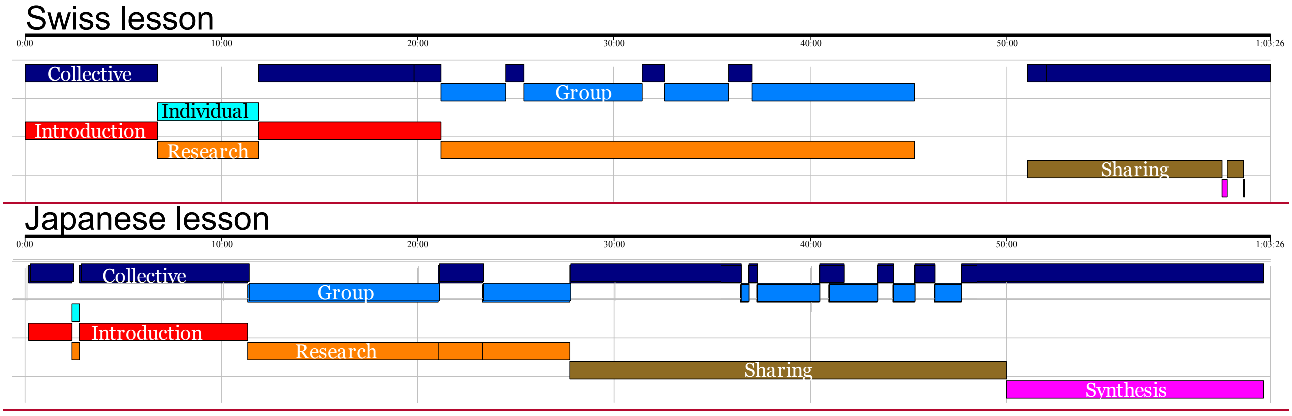

Figure 2. Structure of the two lessons

The teacher's validation of pupil's answers was also one of the biggest differences between the lessons from the two countries. It was the principal and recurring difficulties for Swiss students as well as Japanese students when designing, teaching and discussing the lesson. In the Swiss lesson, the main concern of the Swiss team was to deal with many pupils coming to the teacher during the group research phase (the orange part in Figure 2) to ask him/her: "is this correct?" During this phase, the Swiss teacher takes care of pupils one by one in front of the board and tells the pupils whether or not the solution is correct. The way of validation of the solutions by the Swiss team is an evaluation rather than a validation. In comparison, the Japanese teacher moves from one group to another and asks questions such as "are they really all squares? Could you think about it?", and leaves pupils to make decisions by themselves. This characteristic can be found also in the sharing phase (in bronze colour in Figure 2).

A COMPARATIVE ANALYSIS OF RESOURCES

We carried out a comparative analysis of the resources developed during collaborative design and used while teaching by the two groups of student-teachers, and investigated the cultural factors that yield the differences in teaching practices.

Lesson plans

The lesson plan was a principal resource that the Swiss and Japanese student-teachers developed when designing and implementing a lesson. At the first phase of collaborative design of a lesson, they wrote a draft of lesson plan together; then they finished writing a lesson plan separately in each side and revised it according to the implemented lessons.

In the final versions of lesson plan, one may find some similarities and differences. Both lesson plans include the goals of the lesson, the table showing the chronological progression of classroom teaching, as well as the plan of board writing for which the Swiss team followed the Japanese style lesson plan. Both lesson consists of different phases or moments including introduction, research, sharing, and synthesis. However, one may find several differences in the detail. The Japanese lesson plan, which follows more or less the ordinary format of Japanese lesson plan (e.g. Fernandez & Yoshida, 2001; Miyakawa & Winsløw, 2013), provides a much more precise explanation, not only on the goal and idea of the designed lesson, but also on the objective of the experimentation itself, saying “through the teaching practices with the Swiss task in Japanese school and the analysis of possibilities of the lesson, we also aim to gain a new insight into the development of teaching material”. On the contrary, the table showing the chronological progression of classroom teaching is much more precise in the Swiss lesson plan and provides a precise list of teacher’s actions and pupil’s actions.

One of the critical aspects that yield the differences between the two implemented lessons was the *collectivity* in the classroom activity. The collective dimension is more often referred to in the Japanese lesson plan than in the Swiss lesson plan. While the description of the chronological teaching progression in Japanese lesson plan is short, several remarks are given to promote the collective development of problem solving in the whole classroom setting, for example, in the column of “Teacher’s supports”:

- “When a wrong answer is given, (the teacher) takes it to the whole class, and check why it is wrong, accordingly”;
- “In order to share the succeeded cases in the whole class, make pupils stick the origami on the blackboard”;
- “In order to share pupil’s idea with the whole class, project the origami by the video projector”.

The sense of collectivity in the mathematics classroom, as one may see it in the “structured problem solving lesson” in Japan (Stigler & Hiebert, 1999), is a principal factor that shapes the structure of the Japanese lesson, which explains the relative long time spent on *neriage* or sharing, compared to Swiss lessons (see Figure 2), as well as the teacher’s way of validating pupil’s answer, not individually but rather collectively.

National or regional curricula

When writing the lesson plan, both teams referred to the national or regional curricula. The goals of the lesson in the Swiss lesson plan were cited from the regional curriculum (Table 1). *MSN* in the table means “Mathématique et Science de la nature” (Mathematics and natural sciences), and the numbers show the cycle (first digit) and the domain (second digit; 1: space; 5: modelling). Table 1 includes the goals related to solving geometric problems.

- elements for problems solving (MSN21 in connection with MSN25): solving geometric problems
    - ... by imagining and by using visual representations (codes, diagrams, graphics, tables, ...)
    - ... by sorting and organizing data
    - ... by communicating its results and interpretations
    - ... by asking questions and by defining a framework of study

o ... by mobilizing ... mathematical tools • MSN 21 o Pose and solve problems to structure the plan ▪ ... by representing plane figures ... using sketch ... o Decomposition of a plan surface into elementary surfaces, and recomposition

Table 1. The goals of lesson given in the Swiss lesson plan

In contrast, the Japanese team situated this lesson not in the domain of geometry, but in the domain of function (relation of numerical quantities), in which pupils are required to identify the pattern behind quantity changes. The goal given in the Japanese lesson plan was "Be able to discover different ways to divide a square, identify their pattern, and apply it".

This choice of goal is due to the goals of mathematics teaching given in the national curriculum. The goal of Grade 4 geometry in the Japanese national curriculum is more specific to some geometrical concepts and does not conform to the problem chosen in the project. In fact, the Japanese national curriculum published in 2009 says:

> "Through the analysis of geometrical figures by focusing on their components and their positions, be able to understand the plane figures such as the parallelogram and the rhombus, as well as the solid figures such as the rectangular parallelepiped."

This difference of teaching goal would be another reason why the Japanese lesson had the synthesis phase (*matome*), while the Swiss lesson did not. The Japanese student-teachers are always required to teach specific mathematical concepts or methods, through the problem solving activities. They therefore could not attain their goal without having synthesis phase wherein the pattern of increase of the number of squares is summarized and discussed. In contrast, the Swiss team could arrive at their goal without having it, since their objective is mainly to experience the problem solving activities mentioned in Table 1, and not necessarily to learn a specific concept or method.

Textbooks

Another resource the student-teachers used for the collaborative design of a lesson was the textbooks. The activity of the division of squares was taken from a Swiss textbook. One characteristic of this Swiss mathematics textbook (Danalet et al., 1999) is that it consists of a collection of different problems, often without explicit mathematical knowledge to learn (at least for pupils) and without any suggested order to build a teaching sequence. The problem solving activity for the Swiss team is built on the notion of focusing on the process of resolution, not so much on the acquisition of a specific mathematical knowledge. This would be one of the reasons why the lesson implemented by the Swiss student-teacher did not allocate time for the synthesis.

In contrast, a chapter of the Japanese textbook consists of an amalgam of different elements such as problem-situations, summaries of specific mathematical knowledge to learn, and exercises. The Japanese student-teachers tried to make explicit in the lesson plan specific mathematical knowledge as an objective, although they adopted, as a problem-situation of the designed lesson, the one from the Swiss textbook. This is also due to the structure of Japanese problem solving lesson (Stigler & Hiebert, 1999), including summary of mathematical content (synthesis phase), which the Japanese team tried to follow in their lesson. The idea of problem solving for Japanese team remains in the structure of lesson instead of the objective of mathematics teaching.

DISCUSSION AND CONCLUSION

The cross-cultural comparative analysis of resources in addition to the analysis of lessons allow us to elucidate the characteristics of classroom teaching practices (e.g., lesson structure, teacher's instruction) as well as the "theory" (or the cultural factors) that shapes such practices. Such a "theory" is difficult to identify, because it is often shared within the teachers' community without asking its validity. In our case, the curricular documents (the national or regional curricula and the textbooks) highlighted the perspective of mathematics teaching of each country, emphasizing either the mathematical contents (Japan) or the problem solving skills (Switzerland). The lesson plans allowed us to identify the idea of problem solving with respect to the collective, which yields the differences of implemented lessons between the two countries.

References

Chevallard, Y. (2009). *Remarques sur la notion d'infrastructure didactique et sur le rôle des PER*. Paper presented at the Journées Ampère, Lyon. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Infrastructure_didactique_PER.pdf

Clivaz, S., & Miyakawa, T. (to appear). Cultural effects on mathematics lessons: through the international collaborative development of a lesson in two countries. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)*. Utrecht: Freudenthal Institute and ERME.

Danalet, C., Dumas, J.-P., Studer, C., & Villars-Kneubühler, F. (1999). *Mathématiques 4ème année: Livre du maître, livre de l'élève et fichier de l'élève*. Neuchâtel: COROME.

Fernandez, C., & Yoshida, M. (2001). Lesson study as a model for improving teaching. In *Eye of the storm: Promising practices for improving instruction*. Washington, DC: Council For Basic Education.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199–218. doi:10.1007/s10649-008-9159-8

Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). L*esson study research and practice in mathematics education*. New York: Springer.

Miyakawa, T., & Winsløw, C. (2013). Developing mathematics teacher knowledge: the paradidactic infrastructure of "open lesson" in Japan. *Journal of Mathematics Teacher Education, 16*(3), 185–209. doi:10.1007/s10857-013-9236-5

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap. Best ideas from the worlds teachers for improving education in the classroom*. New York: The Free Press.

A CROSS-CULTURAL STUDY ON TEACHERS' USE OF PRINT AND DIGITAL RESOURCES IN SWEDEN, FINLAND, THE USA, AND FLANDERS: SOME METHODOLOGICAL CHALLENGES

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Cross-cultural studies have inherent challenges as researchers from different cultural backgrounds attempt to make sense of similar-seeming material in unfamiliar contexts and communicate seemingly-obvious aspects of their own culture to outsiders (Clarke, 2013; Osborn, 2004). This contribution explores some of the methodological challenges in a cross-cultural study on teachers' use of print and digital resources in four regions: Sweden, Finland, the USA, and Flanders (Belgium). All but one of the seven team members are insiders to one of the four contexts and to different extents outsiders to the other contexts. In order to benefit from insider-outsider perspectives, we designed five tools to develop alignment of insider and outsider lenses. We describe these tools in this contribution.

INTRODUCTION

This contribution relates to a cross-cultural study on elementary teachers' use of print and digital resources to design and enact mathematics instruction in Sweden, Finland, the USA, and Flanders (Belgium). We are interested in how teachers access and have access to various resources, how they use them, the factors that influence their use, and variation within and across cultural context. Toward that end, we have interviewed ten teachers per context and, as a team, we are currently going through a process of making sense of the interviews applying insider and outsider lenses.

We focus here on developing the seven team members' prerequisite understanding, a term used by Andrews (2007) to relate to the alignment of insider and outsider lenses to facilitate a cross-cultural team's growing intersubjectivity. Our expanding process of developing prerequisite understanding required us to step back and develop processes and instruments including addressing language issues, creating case descriptions of individual participants as an early introduction to teachers within contexts, writing descriptions of contexts and curriculum programs, and developing common understandings of teacher interviews through lengthy conversations between insider-outsider pairs. Through these steps, we aim to build a foundation for analysing resource use with the perspective of insiders taking on outsider views and vice versa.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 73–78). Paderborn: Universitätsbibliothek Paderborn.

CROSS-CULTURAL RESEARCH CONDUCTED BY A CROSS-CULTURAL RESEARCH TEAM

Challenges of cross-cultural research can be both obvious and subtle, as there is a need to establish both insider and outsider lenses with sufficient understanding to both value unique aspects of each cultural context and agree on a comprehensive analytical frame for comparison (Clarke, 2013). We have been grappling with challenges inherent to cross-cultural research, and to undertake such a study in a cross-cultural team.

Challenges inherent to cross-cultural research

Our work has been guided by Osborn's (2004) framework of equivalence building and a set of validity-comparability dilemmas outlined by Clarke (2013). These – at times overlapping – challenges are summarized in Table 1.

Osborn's (2004) equivalences based on Warwick and Osherson (1973)
Conceptual equivalence: identifying/developing concepts with equivalent meanings at a deeply contextual level.
Equivalence of measurement: identifying whether concepts have equivalent salience in each context and developing equivalent indicators for the concepts.
Linguistic equivalence: identifying terms that have equivalent meanings to participants and researchers.
Clarke's (2013) validity – comparability dilemma's
Cultural-specificity of cross-cultural codes.
Deciding between inclusive categories to maximize applicability across cultures vs. distinctive categories that capture explanatory detail.
Cultural specificity of cross-cultural evaluation criteria.
Form vs. Function: confusion with forms that appear equivalent but serve different functions in different cultures.
Linguistic preclusion: linguistic norms influence aspects of discourse that go beyond immediate responses.
Omission: Researchers who lack appropriate cultural terms or concepts may omit notice of critical phenomena.
Disconnection: activities and terms are separated from their local meaning.

Table 1. Challenges inherent to cross-cultural research.

Challenges pertinent to the conduct of cross-cultural research by a cross-cultural research team

One primary approach to addressing the abovementioned challenges is the collaboration between knowledgeable members of each cultural context, as well as defined procedures for building intersubjectivity (Andrews, 2007; Duijker & Rokkan, 1954; Osborn, 2004). We understand the researchers on the Math 3C team as each holding culturally formed backgrounds in studying resource use in one or more contexts. Crucial, however, is a sense of *prerequisite intersubjectivity* among team members (Andrews, 2007). Andrew's (2007) reflection on the functioning of a cross-cultural team of researchers analysing mathematics teaching in five European countries, reveals that although at the beginning of the project their team assumed a shared understanding on central concepts, it took several months to develop prerequisite intersubjectivity and to make significant progress. We were faced with similar challenges and will describe in the following section how we address them.

DEVELOPING THE TEAM'S PREREQUISITE UNDERSTANDING: A NECESSITY FOR GROWING INTERSUBJECTIVITY

We interviewed ten teachers in Finland, the U.S., Flanders, and Sweden in fall 2017 and again in spring 2018 on their use of resources when planning and teaching mathematics (Note: In Sweden and the U.S., one teacher was unavailable for the second interview; in Finland, nine instead of ten teachers have been interviewed). The first interview was more general and addressed teacher and school backgrounds, what resources teachers used, teachers' views on the curriculum resources being used, and teachers' general beliefs on teaching and learning mathematics. The second interview focused in more detail on teachers' actual use of both print and digital resources, centred around a walk-through of planning, decisions, and enactment of a lesson that the teacher taught recently.

Our first step in analysing the interviews was transcribing the audio files in their native languages. Next, we created a summary table for analytical comparison, summarizing in English language teachers' responses to each question of interview 1. We have also developed a low-inference set of codes that arose largely from the structure of the interviews. Throughout this process and related group conversations, it appeared that outsiders lacked knowledge of the other educational contexts and that insiders took aspects of their context for granted, which hindered the team to apply outsider and insider lenses in a useful way. To enable the team to develop its intersubjectivity, we gradually came to develop five tools (See Figure 1). Each of these tools is described below. Some of this work is still ongoing.

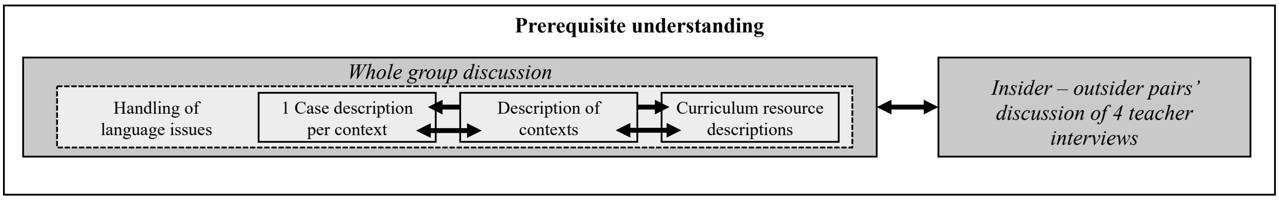

Figure 1. Five tools to develop the research team's prerequisite understanding.

Development of cases

We have developed a set of four descriptive cases of individual teachers, one from each context. Each case was prepared by a team member who is a cultural insider and native speaker of the language (usually the interviewer), but written in English for shared use. We made analytical statements about the teacher's use of resources, checking for confirming and disconfirming evidence in both interviews. Each case also includes images of the print and digital curriculum resources and illustrative quotes, with the intent of providing a full picture of the teacher's decisions.

An important step in the process of developing the cases was full-team review and discussion of them. In fact, we arrived at a common structure and approach through incremental development, review, and discussion, similar to the advocated "joint-development-concurrent" approach by Duijker and Rokkan (1954) and Osborn (2004). The common case structure includes: 1) Teacher education and teaching background, 2) Information about school and class, 3) Information about the selection process of resources, 4) Use of resources and the purposes for use, 5) Teacher beliefs and conceptions, 6) Reported changes in resource use.

The development and discussion of cases played several important roles in our cross-cultural analytical process. It allowed us to undertake initial, low-inference analysis by a cultural insider and make it available to the entire research team. Reading and discussing one another's cases allowed us to ask clarifying questions about the context, the resources, and the teachers' uses of them and consider similar or comparable elements in our own cases. This process allowed us to identify missing elements and uncover additional insider assumptions (Andrews, 2007).

Going through this process surfaced the necessity of composing context descriptions for each of the four contexts and to embark on an approach to handle linguistic challenges (Clarke, 2013; Osborn, 2004). We describe these approaches below.

Context descriptions

The process of writing and reading cases made us aware that significant insider knowledge was necessary to make sense of them. We began writing prefaces to the cases to explain the general educational context of each culture, but then realized that these contextual descriptions were so valuable that they deserved their own focus, methods, and structure in the research process. The list of categories and types of information included underwent several rounds of elaboration as we read each other's drafts, which prompted us to return to our own documents and add more information.

This process led to the discovery of many insider assumptions that we had been unintentionally assuming were universal, a step toward prerequisite intersubjectivity. For example, we did not realize until this process that teachers in some regions typically teach the same grade every year with new students, while in other regions they teach the same students for two or three years with content that proceeds through several grades. Other important differences include teacher autonomy with respect to curricular decision making, and policy decisions.

We eventually settled on the following seven categories: 1) school system-structure, 2) pathways into teaching elementary mathematics, 3) school environment, 4) financial resources for organizing education, 5) decision-making mechanisms in schools in relation to mathematics education (including the selection of instructional resources), 6) student assessment, and 7) monitoring and quality assurance of education. Once we have completed the context descriptions to follow the above categories, we will share them with expert insiders outside of the research team for a final review.

Handling of language issues

The team communicates in English, as all team members have mastery of English. That said, we are learning that commonly used terms have different meanings. For instance, words that are seemingly straightforward in their translation, such as, "instruction," have different meanings when translated into different languages or within the same lanuague used in different contexts. One U.S. team member used the term instruction to refer to what was happening throughout a lesson under the teacher's guidance and orchestration. A Swedish member of the team translated this term into the Swedish as Genomgång, which refers to the part of a lesson when the teacher goes over or reviews material. We have discovered through our discussions that the English tendency to create nouns out of the gerund forms of verbs (e.g. teaching from teach), is not common to other languages, especially Swedish and Finnish, which may use words with separate roots and meanings for the noun and verb forms of a practice. In general, we have found that terms like teaching and learning do not always translate with similar meanings.

To address this challenge, we have begun a multilingual glossary of key terms used by the research team. Each member will identify central terms related to teaching mathematics in their own language and describe their meaning using English languages. We will continue to add to this document and engage in ongoing discussion about their meanings. We have also agreed that we will not limit ourselves to English terms. It is possible, even likely, that some meanings we wish to express cannot be captured in the English language. For instance, the Russian term, *obuchenie* does not have an accurate English translation. The Finnish term *opetus* and Swedish term *undervisning* have similar meanings to the Russian term. Both refer to an activity or interaction in which teachers and students are joint participants and are sometimes translated to English as *learning* and other times as *instruction*.

Descriptions of curriculum resources

We are developing a framework for analysing the printed curriculum resources used by teachers in the study. The development of this framework is ongoing and has been tried out for one Finnish curriculum resource. The framework includes a background and development section, addressing issues such as when, how, and by whom the resource is developed; to what extent it is designed to follow standards; and the stated philosophy. A curriculum resource description section describes the resource's student-facing and teacher-facing components, as well as related digital resources. We also include one complete original lesson from the teacher's guide, which we added with English descriptions of specific lesson components to allow outsiders developing a concrete picture of one sample lesson.

Insider-outsider pairs' discussions of four interviews

Cross-cultural pairs, comprised of a team member who is a clear cultural insider and another who, in all but one case, speaks the language are well in the process of coding four interviews of two teachers (two interviews per teacher). Based on the coding, each pair composes assertions on the teachers' use of resources (a description of the process of creating these assertions goes beyond the purposes of this contribution).

For the first teacher, the cultural outsider for each pair takes a lead on the analysis and creates a set of assertions – an approach also undertaken in Osborn (2004) –, which is reviewed by the insider and discussed by both. These roles change for the analysis of the interview with the second teacher. This process helps toward a better alignment of insider-outsider lenses and we are currently exploring how to share developed understandings across cross-cultural pairs.

Additional information

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References

Andrews, P. (2007). Negotiating meaning in cross-national studies of mathematics teaching: Kissing frogs to find princes. *Comparative Education, 43*(4), 489–509.

Clarke, D. (2013). *Cultural studies in mathematics education*. Paper presented at the Eighth Congress of the European Society for Research in Mathematics Education, Antalya, Turkey.

Duijker, H., & Rokkan, S. (1954). Organizational aspects of cross-national social research. *Journal of Social Issues, 10*(4), 8–24.

Osborn, M. (2004). New methodologies for comparative research? Establishing 'constants' and 'contexts' in educational experience. *Oxford Review of Education, 30*(2), 265–285.

Warwick, D. P., & Osherson, S. (1973). *Comparative research methods.*

NOTES FROM DESIGNING ONE RESOURCE FOR TEACHERS' USE ACROSS THE CONTEXTS OF MEXICO, AUSTRALIA, AND SOUTH AFRICA

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I reflect on design research work with a resource—an instructional sequence on 'Fraction as Measure'—across small-scale studies conducted with teachers and teacher-researchers in Mexico, Australia, and South Africa. The resource at the center of this reflection was designed to support teacher learning and classroom practice, and it has been initially developed and trialled in multiple classroom design experiments in Mexico. By making adaptations to the resource as it is used in different cultural contexts, the collaborating researchers aim to make visible the functions that various design features of the resource serve. I also note how institutional contexts in which participating teachers work profoundly shape studies that can (and those that cannot) be conducted in these contexts.

Teachers' use of resources in planning and teaching varies greatly from classroom to classroom and from country to country. Comparing and contrasting teachers' work across different contexts tends to yield useful theoretical and pragmatic insights. The TIMSS classroom video studies brought attention to the different purposes that can be pursued in teaching of the same mathematical topics and the differences the specific choices and traditions in teaching can have on what students learn (e.g., Stigler, Fernandez, & Yoshida, 1996). Rather than pursuing comparative or observational studies myself, I have been involved in interventionist, design research studies. In those, in order to understand how learning processes can occur in classrooms (e.g., how can a classroom of students learn fractions well), or for teachers (e.g., how can teachers learn to use innovative resources productively), part of researchers' job is to generate the conditions under which such learning will occur.

The nature of the research I pursue is highly collaborative. It also uses comparing and contrasting different enactments of the 'same' learning situation in different settings as one of the basic heuristics in uncovering which parts of learning processes (and thus ways to support them) are largely invariant across learning contexts, and which differ (and thus require different—or additional—means of support). Given that publishers and policy makers often act on the assumption that resources developed in one type of social and cultural context will be suitable and seamlessly adaptable for purposes of another, it may be important to understand which resources, and under what conditions, are adaptable across contexts in ways that result in ambitious and equitable mathematical teaching and learning.

My reflections on instructional resources relate to two types of design research studies through which the resources we design get constituted. First, they relate to explorations

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of student classroom learning, where the resources (i.e., means of supporting that learning) are developed, tested, and revised. Second, they relate to explorations of the learning of teachers who adapt the novel resources to the needs of their classrooms, and in the process help us understand how to refine the resources so that they better support teahcers' use. This paper is my attempt at addressing (variations of) the questions that were proposed as a guidance for this symposium, and, inevitably, it is a patchwork of topics some of which could be useful to pursue in future. These are the variations of questions that guided my reflections:

- What can cross-cultural studies on resource development and use offer the field, in relation to the viability of the products of design research?
- How are design research methodology and the resulting designed mathematics teaching resources illuminated when design research is undertaken and tested in different cultural settings?
- What methodological challenges emerge when undertaking design research related to resource use across different cultural contexts?

WHY DESIGN RESEARCH?

My commitment—and that of my close collaborators—to design research as a way of knowing stems from the power of classroom learning that it makes possible. Such learning is often beyond belief to those who had spent some years teaching children mathematics. Once, as a teacher with four years of classroom experience, I too remained very sceptical when I was told that the aim *as well as the result* of statistics classroom design experiments (Cobb, 1999) was that all students in the classroom learned to reason, in rather sophisticated ways, with measures of centre and spread as they compared statistical distributions. Data convinced me that this was the case.

In spite of the ambitious learning the classroom design experiments can produce, the complexity of the resources by which that learning is usually supported attracts its share of criticism: "Which teachers will be able to use such resources? How will the teachers come to learn to use them? Which teachers will have the conditions set up for required learning? *These kinds of conditions* simply do not occur in most schools." We have argued elsewhere (Visnovska & Cobb, 2019) that it is not helpful to view the complexity of conditions and means of support, required for reproduction of ambitious learning and teaching in new settings, as a limitation of learning resources. Doing so is similar to stating that the need for non-flamable materials is a *limitation* of rocket flight. It turns out to be much more useful to consider the missing materials to be a design problem for which we need to find a solution. In a similar manner, how to establish the (currently missing) conditions under which ambitious teacher learning and teaching can occur should be considerd a design problem well worth addressing.

Addressing this problem would require that we understand a range of suitable conditions for teacher learning and their productive resource use, especially if such conditions are not currently occurring in situ. Here is where the differences in institutional and policy environments in different countries (which shape greatly the conditions of teachers' work and their learning) would need to be taken in consideration when designing the 'existence proof' cases of teacher learning.

'FRACTION AS MEASURE' STORY: CULTURAL CONTEXTS IN DESIGNING FOR STUDENT LEARNING

The impetus for our research program on fraction learning (Cortina, Višňovská, & Zúñiga, 2014) can be traced back to early 2000's when, working on his dissertation on instructional design in ratio in a seventh grade US classroom (Cortina, 2006), Jose noticed that the only student who was not progressing in his learning had not previously developed even the basic quantitative sense of fractions. While this student was an exception in his US classroom, his mathematical background—Jose was aware—was typical in many classrooms in Mexico. It was indeed typical of classrooms where most of the world children get to learn mathematics. In order to understand how to support students in any classroom in learning to reason with ratios, the instructional sequence would have to first attend to supporting their development of a quantitative notion of fraction.

If we tried to develop the fraction precursor to the ratio sequence in a US classroom similar to one in the ratio experiment, we would likely have struggled to negotiate meaning between the majority of the students who already had the precursor fraction insights, and the few of those who did not. This would especially be the case because we were, at that point, not clear about how the development of quantitative meaning of fraction can be proactively supported at a classroom level.

As instructional designers, we were keenly aware that if most students had already developed a substantial mathematical idea outside of the classroom design experiment situations, they contribute this idea 'for free', even when the designed resources are insufficient for supporting their reasoning about it. In such situations, even though the students learn successfully, we *do not learn* about what made it possible for this mathematical idea to emerge in classroom discussions, or what would be essential for the teacher to understand and do when supporting such development in another classroom. We thus conducted the initial fractions classroom design experiments in Mexico (and in lower grades), aiming to reduce the chance that the insufficiencies of the design would be masked by students' insights developed elsewhere. This illustrates one of the ways in which purposeful selection of socio-cultural context can play an important role in design research studies.

Reseaching across socio-cultural context also plays a role in allowing to demonstrate the robustness of developed means of supporting student learning. In 2017, researchers in the South African Numeracy Chair Project (SANCP) adopted our 'fraction as measure' resources for purposes of (1) trialling whether these would adequately support initial fraction learning in their context, and if so, for (2) supporting the learning of teachers in grades 4-7, in the research and development approach to local improvement and capacity-building efforts. To gauge the viability of fraction resources for the local teaching needs, SANCP researchers first conducted small-scale trials where they used our resources in working with students in an after school Maths Club. They followed this by a one-week implementation study in three grade 3 classrooms, with 105 students, taught by Pam– a researcher-teacher who was herself learning to use the resources (Vale & Graven, 2018). In Mexico, Guadalupe, a teacher completing her Masters degree, acted as a researcher in her grade 5 classroom, conducting a classroom design experiment, in weekly 35 min lessons, for 18 weeks duration. The differences in how these studies were organised, and in adaptations made, were developed in response to the specific affordances and constraints of each setting (Cortina, Visnovska, Graven, & Vale, 2019). However, in both contexts, documented learning was beyond what is typically expected. While the biggest gains were in terms of students' reasoning, the crude pre/post correctness on tasks measures showed improvement from 8% to 71% success rate on 3 unit fraction comparison tasks in South Africa (N = 83), and from 20% on the same 3 tasks to 100% on 9 more complex unit fraction comparison tasks in Mexico (N = 20). These results were also consistent with learning in Australian grade 3 and 4 classrooms, where data on student learning were collected and analysed for purposes of teachers' learning. In this context, we did not seek permisions for reporting student data as a means of increasing our chances of gaining access to schools and teachers.

With regards to the methodological challenges we encountered while working across cultural settings, a number of these were related to 'translation' of the designed resources, both between languages, and in adjusting for differences in student competence in the language of instruction. These translation challenges typically led to enhanced understanding of the instructional sequence, in addition to its improved local design. They also provided opportunities for clarification of specific ideas that form the design rationale, among the different researchers and teachers involved.

Specifically, in South African classrooms, year 3 students are for the first time studying in English or Afrikaans, while over 90% of them speak different language at home. *Fraction as Measure* sequence is presented through a story of how a group of ancient peoples measured length. The South African adaptations included using props, miming the actions of the main characters, and including a student to act as the child character in the story. While we used these techniques in storytelling in both Mexico and in Australia previously, South African collaboration highlighted these as key components in supporting equitable learning in linguistically diverse classrooms.

We have also collaborated on developing a picture storybook version of the story (Vale, Graven, Visnovska, & Ford, 2019) for classroom uses where spoken word would be additionally complemented with both images and written text. Once the storybook was developed (in English), its translations to different languages in which we taught (Spanish, Afrikaans, and isiXhosa) was followed by additional translations (to Indonesian and Slovak) as the storybook became a useful tool for introducing some of the key means of supporting learning upon which the instructional sequence was built. One of such means of support is an avoidance of use of standard fraction language (e.g., *one half*) and notation (1/2), as the meanings that typically accompany these do not involve quantitative comparisons (e.g., half is when something is broken into two pieces). For this reason, the sequence uses made up terminology (e.g., *small of two*) while students develop quantitative meanings of fractions. This terminology is replaced with standard mathematical vocabulary at a later point.

The languages of our translations happen to represent distinct language families and branches, with some of them ‘missing’ the grammar features we tend to take for granted. The ways in which language adaptations needed to be made to maintain the conceptual focus of the resource—and clarifications of the purposes of different resource features, such as why do we need to make up new terminology for unit fractions—served as a learning resource for involved teachers and researchers. It is notable that English speakers tend to undervalue the support that the made up terminology can provide in classrooms, as fewer opportunities typically arise for directly discussing and clarifying purposes of terminology. The context of collaborating on translations both provided such opportunities, and made us aware of their importance in learning about the resources. Developing specific professional development activities that would engage the teachers in clarifying the role of language within the resource is of quite some importance.

DESIGNING TEACHERS’ RESOURCES

While it is possible for classroom design research to produce resources for supporting students’ learning directly, our goal is to instead design *teachers’ resources*. As such, we aim to understand the features of the resource design that (1) facilitate teachers’ access to and familiarisation with the rationales that underpin the resource, and those that (2) support the deliberate adaptation of the resource to the specific circumstance of the classroom, and thus support teacher’s in-class use. Here, too, cross-cultural differences play an important role, both in whether different types of studies can be conducted in different institutional contexts, and in the types of support the teachers would need in order to develop increasing competence in use of the new resources.

In Mexico, Master students like Guadalupe, who also work as classroom teachers, can conduct slow paced classroom design experiments in their own classrooms, spanning many weeks. In contrast, similar freedom to depart from the structure and timing of the designated curriculum (Remillard & Heck, 2014) is rarely (if ever) possible in Queensland schools. Teachers can only trial fraction resources in the weeks when fractions are officially the focus in the curriculum, which requires fast pacing through the sequence (changes happen daily), exaggerated support from the researcher (researcher-taught and teacher-observed lessons or co-teaching, with a short debrief or planning meeting), and does not provide the teachers with enough time to trial, reflect, rethink, or to come to terms with the mathematical ideas, why they matter, and how to support their emergence in the classroom.

To conclude, I aimed to outline some ways in which design research on and with resources generates useful insights when conducted across different cultural settings and national borders, and which I hope will contribute to the symposium discussions.

References

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*(1), 5–43.

Cortina, J. L. (2006). *Instructional design in ratio.* (Dissertation), Vanderbilt University, Nashville, TN.

Cortina, J. L., Visnovska, J., Graven, M., & Vale, P. (2019). Instructional design in pursuing equity: The case of the ‘fraction as measure’ sequence. In N. Govender, R. Mudaly, T. Mthethwa, & A. Singh-Pillay (Eds.), *Proceedings of the 27th annual conference of the SAARMSTE* (pp. 16–28). Durban, South Africa: SAARMSTE.

Cortina, J. L., Višňovská, J., & Zúñiga, C. (2014). Unit fractions in the context of proportionality: supporting students' reasoning about the inverse order relationship. *Mathematics Education Research Journal, 26*(1), 79–99.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46*(5), 705–718.

Stigler, J. W., Fernandez, C., & Yoshida, M. (1996). Traditions of school mathematics in Japanese and American elementary classrooms. In P. Nesher, L. P. Steffe, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning.* Hillside, NJ: Lawrence Erlbaum.

Vale, P., & Graven, M. (2018). Facilitating conceptual engagement with fractions through suspending the use of mathematical terminology. *Proceedings of the 42nd Conference of the IGPME.* Sweden: PME.

Vale, P., Graven, M., Visnovska, J., & Ford, C. (2019). *Mama Khanyi and the pots.* Makhanda, South Africa: SANCP, Rhodes University.

Visnovska, J., & Cobb, P. (2019). Supporting shifts in teachers’ views of a classroom statistical activity: problem context in teaching statistics. *Mathematical Thinking and Learning.* doi:10.1080/10986065.2019.1576003

Symposium C: Assessment tools in support of teachers' curricular decision making

ASSESSMENT TOOLS IN SUPPORT OF TEACHERS' CURRICULAR DECISION MAKING

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The interactions of teachers and learning resources has been widely studied in the last decades. As assessment resources are evolving alongside with technological developments, they are now included either as supplements or as an integral part of the corpus of learning resources. This symposium will demonstrate interaction with student work from two different projects from the USA and Israel: On Going Assessment Project (OGAP) and Seeing the Entire Picture (STEP). Each one of the teams will present its own perspective on teachers' curricular decision making focusing on the incorporation of assessment materials into the regular mathematics program taught in class.

RATIONALE

Textbooks and other learning resources, and the ways teachers interact with them, have been the focus of many studies in the past decades. Assessment resources, while at times not part of the core learning resources, play a major role in the teaching and learning process. Nowadays, assessment resources are often included either as supplements or as an integral part of the corpus of learning resources. Given the pace at which technology evolves, there is increasing potential for technological tools to further enhance the process of assessment.

This symposium focuses on two different projects from the USA and Israel: Ongoing Assessment Project (OGAP) and Seeing the Entire Picture (STEP). Both of these projects are supplemental to the primary curriculum resources and provide opportunities for teachers to analyze and interact with student work, which in turn inform their instructional decisions. Using these resources requires teachers to integrate two types of resources in relation to curricular goals. The two different projects provide two perspectives on different and complimenting aspects of teachers' curricular decision making when using assessment resources. We focus on the common challenge of making assessment materials work alongside a regular math program taught in class.

ASSESSMENT PROJECTS

Ongoing Assessment Project

The Ongoing Assessment Project (OGAP) originated in 2003 by mathematics educators from Vermont, USA. Based on research on student learning, OGAP offers teachers a set of formative assessment tools, resources, and routines to help them systematically and continuously respond to student understanding in relation to learning trajectories in core mathematics domains (Petit, Hulbert and Laird, 2017). The

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 87–88). Paderborn: Universitätsbibliothek Paderborn.

assessment process is based on an assess-analyze-respond cycle. *OGAP Frameworks* synthesize problem contexts, structures, and learning trajectories, which teachers use to analyze student work and determine instructional next steps. Evidence from recent research indicates that use of OGAP leads to significant gains in students' problem-solving accuracy and strategy use and teachers' understanding of student thinking (Supovitz, Ebby, Remillard, & Nathenson, 2018).

Seeing the entire picture project

Seeing The Entire Picture (STEP) is an online mathematics formative assessment platform (Olsher, Yerushalmy, & Chazan, 2016). STEP provides automatic assessment of student answers, typically to open ended rich tasks that require the student to construct an example related to different mathematical claims, which are referred to as Example Eliciting Tasks (EET). STEP EET are designed to have many possible correct answers, which could be automatically analyzed for different characteristics. Along with characteristics related to research or practice based mistakes, the answers could also be analyzed according to different characteristics of the mathematical objects comprising the answer, and the methods used in constructing these objects (heuristics). STEP provides the teacher with real-time domain specific learning analytics on a student and classroom level (Olsher & Aby-Raya, Accepted).

Assessment tools in support of curricular decision making

The two above described projects focus on using student work as evidence, and different ways to connect it to the mathematics program taught in class. The main goal of this symposium is to elicit the challenges teachers face in the process of using these assessment tools in support of curricular decision making, and the lessons that we can learn about design of assessment tools and resources.

References

Petit, M., Hulbert, E, & Laird, R. (2017). *OGAP Frameworks*. Retrieved from https://ogapmathllc.com/resources-and-related-publications/

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How Might the Use of Technology in Formative Assessment Support Changes in Mathematics Teaching? *For the Learning of Mathematics, 36(3)*, 11–18.

Olsher, S., & Abu-Raya, K. (Accepted). Teacher's attention to characteristics of Parabola sketches: differences between use of manual and automated analysis. To be presented at *ICTMT 14* (Essen, Germany, July 2019).

Supovitz, J. A., Ebby, C. B., Remillard, J., & Nathenson, R. A. (2018). Experimental Impacts of the Ongoing Assessment Project on Teachers and Students. *CPRE Research Reports.* Retrieved from https://repository.upenn.edu/cpre_researchreports/107

Symposium D: Digital platforms and mathematics teachers' documentation work

DIGITAL PLATFORMS AND MATHEMATICS TEACHERS' DOCUMENTATION WORK

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ANALYSING DIGITAL PLATFORMS AND THEIR USE

In many countries nowadays, different kinds of digital platforms are available for mathematics teachers. Some of these platforms are proposed and promoted by the "official" institution (ministry of education, for example); they often offer a highly structured content that is expected to support teaching practices aligned with the official expectations. Others are designed as open repositories, where teachers can build their own collections of teaching resources, and share contents with colleagues and with their students.

In this symposium we retain the framework of the documentational approach (Gueudet, Pepin, & Trouche, 2012, Trouche, Gueudet, & Pepin, 2019) to study the consequences, actual and potential, of the use of digital platforms on mathematics teachers' work. Mathematics teachers look for resources (Adler, 2000), choose resources, modify them and use them in class. These resources can be traditional textbooks, digital resources but also students' productions. Along this documentation work, teachers develop documents associating resources and a scheme of use for these resources. The features of the resources influence this development, in an instrumentation process; at the same time, teachers modify the resources they use according to their professional knowledge, in an instrumentalisation process. Teachers develop structured document and resource systems (Trouche, Gueudet, & Pepin, 2019). Communities of teachers working together can also develop collective resource systems (Gueudet, Pepin, & Trouche, 2013).

The concepts of documentation work, documents and resource systems can inform how digital platforms potentially can (or actually do) transform teachers' work. More generally, we claim that a 'resource' approach in mathematics education is needed, in a context where digital resources, including platforms, are widely available.

In previous works (e.g. Gueudet, Pepin, Sabra, Restrepo, & Trouche, 2018; Pepin, Gueudet, Yerushalmy, Trouche, & Chazan, 2016) we have suggested to analyse e-textbooks in terms of their interactivity and 'connectivity', and we have emphasized the need to consider how these e-textbooks are, or can be, connected with teachers' resource systems. These notions are also relevant for analysing digital platforms, and for understanding how digital platforms link to and are likely to modify teachers' work. Is it possible for teachers to import elements of their own resource system into the platform? Or to import resources from the platform into their resource system? To share resources with colleagues, or with students on the

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platform? What are the features of such platforms, and do they influence teachers' documentation work, in particular platforms proposed by the institution, and do they foster instrumentation processes? In Gueudet, Pepin, Courtney, Kock, Misfeldt, and Tamborg (to appear) we compared three digital platforms (in Denmark, France and The Netherlands). We discussed their affordances and constraints; the reasons explaining the differences observed, and the effects in terms of teachers' documentation work.

In this symposium we further study and discuss these issues, by drawing on four contrasting national cases. [Case 1] In Denmark digital platforms are compulsory and propose a particular interpretation of the curriculum. Nevertheless, the teachers interpret the mathematics content in their own documentation work. [Case 2] In France, we study how different kinds of platforms offering individual resources, and also tools to associate these resources, can interact with the development of teachers' resources system. [Case 3] In the Netherlands we analyse the affordances and constraints of the most commonly used not-for-profit platforms for mathematics teachers (primary and secondary teachers) in terms of their documentation work. [Case 4] In the US, the Ohio Mathematics Teacher Hubs Project promotes collective work for mathematics teachers and mathematics intervention specialists, who can discuss and share resources.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education, *Journal of Mathematics Teacher Education* 3, 205–224.

Gueudet, G., Pepin, B., Courtney, S., Kock, Z.-J., Misfeldt, M., & Tamborg, A. (to appear). Digital platforms for teacher documentation work: Affordances and constraints for teacher curriculum design. In A. Clark-Wilson et al. (Eds.), *Mathematics Education in the Digital Age: Learning, Practice and Theory.* London: Routledge.

Gueudet, G., Pepin, B., Sabra, H., Restrepo, A., & Trouche, L. (2018). E-textbooks and connectivity: proposing an analytical framework. *International Journal for Science and Mathematics Education*, *16*(3), 539–558.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.) (2012). *From textbooks to 'lived' resources: Mathematics curriculum materials and teacher development*. New York: Springer.

Gueudet, G., Pepin, B., & Trouche, L. (2013). Collective work with resources: An essential dimension for teacher documentation. *ZDM Mathematics Education*, *45*(7), 1003–1016.

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. (2016) E-textbooks in/for teaching and learning mathematics: A disruptive and potentially transformative educational technology. In L. English & D. Kirshner (Eds.) *Handbook of Research in Mathematics Education*, 3rd Edition (p. 636–661). Taylor & Francis.

Trouche, L., Gueudet, G., & Pepin, B. (2019). *The 'resource' approach to mathematics education*. Cham: Springer.

DIGITAL SHARING AND LEARNING SPACES: THE OHIO MATHEMATICS TEACHER HUBS PROJECT

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In the U.S., professional learning and sharing opportunities that are flexible, free, meaningful, and focused on mathematics teaching and learning are particularly difficult to find for mathematics teachers working in rural school districts. In this report, I share results from the first seven months of a project designed to provide a digital space for grades 6 to 12 mathematics teachers and math intervention specialists from rural schools to discuss issues and share ideas with colleagues from other schools and districts and university mathematics education faculty. Specific factors contributing to an extended recruitment period for participating teachers are discussed, along with details of the project's next phase.

INTRODUCTION

Digital platforms permeate grades K-12 and post-secondary education, supporting both mathematics learners, through video tutorials and online adaptive math activities and games, and teachers as they prepare lessons, select tasks, assess student understandings and skills, and collaborate with colleagues. Digital education platforms not only play a role in teaching and student learning, but also teachers' professional learning through MOOCs and online communities. For example, the *Math MOOC UniTo* project provides professional learning for Italian mathematics teachers through MOOCs delivered on a Moodle platform (Taranto, Arzarello, & Robutti, 2018). The National Centre for Excellence in the Teaching of Mathematics (NCETM) is an online platform funded by the ministerial Department for Education in England that provides professional development courses for mathematics teachers, innovative resources and tools, hands-on research projects, and various online communities (NCETM, Community section, n.d.). In the U.S., professional learning and sharing opportunities that are flexible, free, meaningful, and focused on mathematics teaching and learning are difficult to find. For mathematics teachers working in rural school districts, such collaborative professional learning opportunities are particularly limited. The *Ohio Mathematics Teacher Hubs Project* provides a digital space for grades 6 to12 (student ages 11 to 18 years) mathematics teachers and math intervention specialists (special education teachers) from rural schools in Ohio (a midwestern U.S. state) to discuss issues and share ideas with colleagues from other schools and districts and university mathematics education faculty. In this report, I share results from the first seven months of the project. The project uses a design-based approach (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to design and explore the question: How can grades 6 to 12 mathematics teachers, math intervention specialists, and university

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mathematics education researchers work collaboratively online to support rural students' college and career readiness in mathematics?

BACKGROUND

The rationale for focusing on rural school districts derives from several factors. Rural schools make up a significant proportion of all public elementary and secondary schools by typology in the U.S., but rural districts receive a small percent of state education funding (Glander, 2017). In addition, rural schools and communities face considerable challenges with high poverty rates, students with special needs, and recruiting and retaining highly qualified teachers, particularly in high-needs areas such as mathematics (Monk, 2007). The midwestern state of Ohio has the fourth largest rural student population in the U.S. and reflects several national rural characteristics. Although 37.7% of public school districts in Ohio are rural, only 18.7% of state funds are appropriated to rural districts (Ohio Department of Education, 2019). In addition, rural districts have a high average percent of inexperienced (i.e. first- or second-year) teachers (12.7%) and a low average percent of teachers with 10 or more years of experience (55.9%) compared to suburban districts (7.9% and 60.5%, respectively) (Ohio Department of Education, 2019). Rural districts in Ohio face academic challenges as well. The average percent of students scoring at or above 'Proficient' on the state 2017-2018 mathematics achievement tests decreased at each grade level or course for rural districts from grade 6 through grade 8 and again from Algebra 1 to Geometry. Finally, mathematics teachers in rural districts frequently work in buildings where they are the sole grade-level or course teacher in that building, and, possibly, the entire district. As such, opportunities to collaborate, co-plan, and connect with peers who teach the same grade level or course are rare.

METHODOLOGY

The project takes as foundational the assertion that teachers learn when choosing, transforming, discussing, implementing, and revising resources (Gueudet, Pepin, & Trouche, 2012). Furthermore, the project utilizes the documentational approach of didactics (Gueudet, Pepin, & Trouche, 2012) to study the consequences, actual and potential, of the use of digital platforms on teachers' work. Central to the documentational approach is documentational genesis, with its dialectical processes involving both a teacher's shaping of the resource and her teaching practice being shaped by the resource (Gueudet, Pepin, & Trouche, 2012). Documentational genesis plays a significant role due to the project's focus on teachers' work on and with resources, both individually and collectively. By 'resources', I refer to curriculum resources, defined by Pepin and Gueudet (2018) as "all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (p. 132). Such resources include: text resources (e.g. textbooks, teacher curricular guidelines, websites, worksheets, syllabi, tests); other material resources (e.g. manipulatives, calculators);

digital-/ICT-based curriculum resources (e.g. interactive e-textbooks) (Pepin & Gueudet, 2018, p. 132).

From its inception, the project has utilized a cyclical design that follows a process similar to that described by Reeves (2006, p. 59): 1) analysis of practical problems by researchers and practitioners in collaboration, 2) development of solutions informed by existing design principles, 3) iterative cycles of testing and refinement of solutions in practice, and 4) reflection to produce 'design principles' and enhance solution implementation. To date, only the initial step in the four step design process has been realized. The initial seven months of the project (i.e. Phase 1) were slowed down with participant recruitment and attempts to generate and compile areas of research interest for anticipated inquiry communities. Specifically, Phase 1 consisted of five online sharing and learning experiences (involving nine teachers and one district curriculum director), and 10 face-to-face (on-site) learning, sharing, and recruitment sessions (involving a total of 83 teachers, four curriculum directors, six building principals, and two district superintendents).

Throughout Phase 1, the project employed webinars (i.e. web-based video conferences that use the internet to connect multiple individuals) through a free web hosting platform (i.e. Google Hangout) concurrent with a free file storage and sharing service (i.e. Google Drive) to interact with participants. Such webinars were free to teachers and avoided requiring teachers register as university students (with a 66 euro general fee) and sign up through the university learning management system for an additional fee. In addition, eight school districts received on-site recruitment meetings (requiring personal travel by researchers up to 4.17 hours and 388 km one-way) with five such meetings involving professional learning sessions requested by the district. Finally, a free web hosting service (i.e. Weebly) made the project website accessible and allowed for the promotion of project webinars, project news and updates, contact information, and an online catalogue of materials for loan (e.g. books, physical manipulatives)—for free—to Ohio public school mathematics teachers and math intervention specialists.

RESULTS

Online and face-to-face discussions emphasized the reality that some teachers working in rural districts and living in small towns or rural areas were unable to access project activities via digital platforms due to broadband availability or affordability issues. Such concerns necessitated flexibility in webinar sessions—ranging from after school sessions, so teachers could use their school's internet, to evening sessions that allowed for at-home participation. The project anticipates that some future activities will demand additional on-site sessions that allow teachers with significant internet connectivity issues to participate locally. The necessity for an extended recruitment period can be accounted for by several factors: district mandated teacher policies, initiatives, and perceptions; potential participant expectations and perceptions of university researcher-teacher partnerships; and university expectations of faculty researchers.

District mandated teacher policies, initiatives, and perceptions

In general, university researchers are dissuaded from contacting teachers directly about their potential participation in projects; rather, schools and districts desire researchers initiate contact through district administration (e.g. superintendent, building principals). Such filters had the potential to limit the number of mathematics teachers and math intervention specialists that received any information about the project. For example, many districts were already committed to time-intensive mandates or initiatives across all grade levels and content areas (e.g. writing across the curriculum, social justice programs). As such, some administrators indicated a weariness to invite their faculty to participate in additional time-intensive projects. Furthermore, when considering projects that focus on supporting students' content-specific (e.g. mathematics) college and career readiness, administrators desire participation from all high school (grades 9-12) and/or middle grades (grades 6-8) content teachers—to potentially impact the most students in the shortest amount of time. Unfortunately, such expectations may require participation from teachers who may not fully believe in or support a project or have time to fully engage in project activities. Therefore, the *Ohio Mathematics Teacher Hubs Project* was designed so that individual 6th to 12th grade teachers could participate—not the entire district. Such flexibility was not always found to be favourable by district administrators. Finally, approaching district administration involved overcoming an additional perception—the belief that district (or teacher) participation was going to be an expense to the district. It was not uncommon for administrators to promptly ask, "How much will this cost the district [financially]?" In order for the project to coalesce into communities of inquiry, outreach to district administrators will need to be clearer and more concise as to district expenses (there are none) and teacher commitments.

Potential participant expectations and perceptions of university researcher-teacher partnerships

In the U.S., participation in research projects is typically something teachers engage in on their own time, without receiving an additional stipend through or time- or instructional (course)-release by their district. Many rural mathematics teachers already have demanding teaching schedules. For example, one Phase 1 discussant indicated she taught Algebra 1 (twice), Applied Mathematics, Pre-Calculus, and Calculus every day (Monday through Friday) and each of her classes except Calculus was inclusive (i.e. general education setting in which students with and without disabilities learn together). Project participation would add another commitment to her overburdened teaching schedule and responsibilities from district initiatives. Pragmatically speaking, this is a lot to ask from teachers with lives of their own. Aside from time constraints, three perceptions became apparent when speaking with teachers during face-to-face recruiting sessions—each suggest that teachers' prior interactions with university faculty (any university, not specifically the university at which I am employed) resulted in one-sided relationships. Firstly, several teachers asked whether the project intended to place pre-service teachers with project participants. Eight teachers explicitly indicated that prior teacher-university faculty interactions resulted in the university requesting placement of a student teacher (for which teachers received little to no compensation). Secondly, several teachers asked if there were courses (which range in cost from 472-1,415 euros) or workshops (range in cost from 270-808 euros) the university required they take as part of their project participation—which teachers typically pay for themselves without district reimbursement. Twelve teachers indicated that prior teacher-university faculty interactions (from various universities) resulted in requests for teachers to register for courses or workshops. Lastly, four teachers indicated a lack of interest in the project due to their own or a colleague's prior experiences, in which teachers did not feel they were an actual partner in the research; rather, the researcher was only interested in using the teacher and/or students as experimental subjects.

University expectation of faculty researchers

Since the state of Ohio eliminated the master's degree requirement for teachers, the percent of full time teachers with at least a master's degree has dropped from 68.2% in 2012-2013 to 64.9% in 2017-2018 (a decrease of over 4,700 teachers). At the university where I am employed, the number of teachers enrolled in master's level mathematics education courses has decreased to the point where courses regularly get cancelled or scarcely make enrolment minimums. As such, the university is cautious of faculty taking time to work with teachers that are not registered students (i.e. paying tuition), unless such work is grant-funded. Therefore, it is a challenge to get research that does not generate tuition or grant funds approved by university administrators, unless such research results in prompt and reliable journal publications.

CONCLUSION

What became clear during Phase 1 is the need for the project to find an external funding source. External funding would: a) allow university researchers to promote the project to districts and teachers as vetted and significant; b) reassure districts that there will be no cost for teachers' professional learning, c) allow for a stipend to be paid to participating teachers, and d) pay faculty researcher and graduate student salaries and benefits. Therefore, the project applied for a three-year $327,000 research education foundation grant involving 24 mathematics teachers and two math intervention specialists (from 20 different rural school districts), one curriculum director (grades 6 to 12), one university mathematics education researcher, and one PhD-level graduate student. Potential participants were self-selected and met the following requirements: a) mathematics teacher or math intervention specialist in any of grades 6 to 12; b) teach in a rural school district in Ohio; c) have an interest in exploring how grade 6 to 12 mathematics teachers, math intervention specialists, and university mathematics education researchers can collaborate in an online community of inquiry to support rural students' college and career readiness in mathematics; and d) available to spend approximately 36 hours of online collaboration over the course of 12 months, with a subset of teachers allocating an additional 10 hours of face-to-face time (onsite at teacher's school). Phase 2 of the project will involve the creation of inquiry communities between researchers and teachers to explore ways of improving learning environments for students in mathematics classrooms. Online and face-to-face interactions with teachers (during Phase 1) generated the following Phase 2 research ideas: 1) effective ways to meet the needs of all students in inclusive mathematics classrooms (in rural schools)—from students with learning challenges to gifted learners; 2) effective ways to assign and evaluate homework in rural mathematics classrooms; 3) effective note taking strategies that support students' mathematical development and promote and support productive homework and assessment study habits in rural mathematics classrooms; 4) effective ways to cover 100% of the content in 80-85% of the academic year (due to timing of state achievement assessments); e) effective ways to incorporate instruction and assessments that focus on the mathematical practices of modelling and reasoning.

References

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences*, *2*(2), 141–178.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Glander, M. (2017). *Selected statistics from the public elementary and secondary education universe: School year 2015–16* (NCES 2018-052). U.S. Department of Education. Washington, DC: National Center for Education Statistics.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.) (2012). *From textbooks to 'lived' resources: mathematics curriculum materials and teacher development*. New York: Springer.

Monk, D. H. (2007). Recruiting and retaining high-quality teachers in rural areas. *Future of Children*, *17*, 155-174.

National Centre for Excellence in the Teaching Mathematics (NCETM). (n.d.). *About the NCETM website*. Retrieved from https://www.ncetm.org.uk/ncetm/

Ohio Department of Education (2019). *Ohio school report cards: download data*. Retrieved from https://reportcard.education.ohio.gov/download

Pepin, B., & Gueudet, G. (2018). Curriculum resources and textbooks in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 132–135). Cham, Switzerland: Springer.

Reeves, T. C. (2006). Design research from a technology perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 52–66). London: Routledge.

Taranto, E., Arzarello, F., & Robutti, O. (2018). MOOC as a resource for teachers' collaboration in educational program. In V. Gitirana, T. Miyakawa, M. Rafalska, S. Soury-Lavergne, & L. Trouche (Eds.), *Proceedings of the Re(s)sources 2018 Conference* (pp. 167–170). Lyon, France: Ecole Normale Supérieure de Lyon.

DIGITAL PLATFORMS FOR TEACHERS' DOCUMENTATION WORK: EXAMPLES IN FRANCE

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In a context of curriculum reforms, two digital platforms have been opened in France (around 2017). CARTOUN allows teachers to share lesson plans; DRSB offers different kinds of resources and tools to design lessons. I analyse and compare these two platforms using a documentational approach perspective, and in particular the concept of connectivity introduced to study e-textbooks. This analysis uncovers intentions of the institution: fostering teachers' agency and collective work, but also shaping their documentation work through instrumentation processes.

In this paper I analyse and compare two digital platforms opened in France around 2017, in a context of curriculum reform. I firstly present the theoretical frame of this study: the documentational approach to didactics, and in particular the concept of connectivity. Then I briefly describe the French context, before analysing and comparing the two platforms.

THEORETICAL PERSPECTIVE

This study refers to the documentational approach to didactics (DAD, Gueudet, Pepin, & Trouche, 2012). This approach focuses on the interactions between teachers and resources, and on the consequences of these interactions. A teacher selects resources, modifies them, uses them in class; this is called the teacher's documentation work. Along this work, the teacher develops what is called a document: recombined resources, and a scheme of use (Vergnaud 1998) of these resources, for a given aim of the teacher's professional activity. This process is called a documentational genesis, it has two strongly linked components. The features of the resources influence the schemes developed by the teacher: this is the instrumentation process. The professional knowledge of the teacher influences his/her choices (modifications of the resource, of its intended use): this is the instrumentalisation process. Hence the affordances and constraints of the resources (here, the platform) contribute to shape the teacher's documentation work, and the teacher's resource systems (developed along their professional activity, Trouche, Pepin, & Gueudet, 2019). For a more precise analysis of the platforms affordances and constraints, I refer to the concept of connectivity (Gueudet, Pepin, Sabra, Restrepo, & Trouche, 2018), introduced in the context of the study of e-textbooks. We defined the connectivity of an e-textbook as its potential in terms of connections: cognitive and also practical for a teacher (or a student) using it. We distinguished between macro-level connectivity, which concern connections between the e-textbook and external content: other websites, resource system of the user, between users etc. This macro-level connectivity does not depend on the

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 99–104). Paderborn: Universitätsbibliothek Paderborn.

mathematics content. Micro-level connectivity concerns a particular mathematical theme and connections within the e-textbook: between different representations, with other disciplines or real-life situations, with a given software, etc.

The research question we study here could be presented as:

"For two selected platforms in France, what is their connectivity, and how does this inform us about their affordances and constraints for teachers' documentation work?"

CONTEXT: DIGITAL PLAFTORMS AND EDUCATIONAL POLICY

In France a new curriculum for primary and lower secondary school (from grades 1 to 9) has been introduced since September 2016 (Gueudet, Bueno-Ravel, Modeste, & Trouche, 2017). This curriculum reform coincided with the so-called "Digital Plan for School", introducing different means to support the use of technology in schools, and led in particular to the design of the two digital platforms studied here.

The first one is called "CARTOUN", meaning "Map of technology use". The CARTOUN platform offers to teachers the possibility to share their lesson plans (initially concerning lessons using technology, but now open to any kind of lesson). The author indicates his/her location, which appears on a dynamic map. Any teacher from primary school to upper secondary can upload a lesson plan. Amongst the different fields to fill when uploading a lesson plan, the teacher gives his/her contact e-mail, and can even invite colleagues to visit his/her class. Indeed the intention of the institution with this platform (opened in 2016) was not only to create a repository of resources designed by teachers, but to foster the development of teachers' networks.

The second one is called the "Digital Resources for School Database" (DRSB in what follows, for more details see Gueudet et al. to appear). In the frame of the "Digital Plan for Schools", the intention of the ministry of Education was to start replacing traditional textbooks on paper by e-textbooks. This led to the design (by private publishers) of the DRSB platform, opened in September 2017. Its use is not compulsory for teachers (a direct use by students is not intended). It is the only "official" platform in France offering resources which cover the whole mathematics curriculum from grades 7 to 9 (called "cycle 4", in the new curriculum).

CONNECTIVITY, AFFORDANCES AND CONSTRAINTS

The CARTOUN platform

The CARTOUN platform offers lesson plans designed by individual teachers, by groups of teachers or groups associating teachers and inspectors. The resources can be found by using a browser, or on the platform dynamic map, where the authors of resources appear (Figure 1). For mathematics at cycle 4, 80 lessons are available.

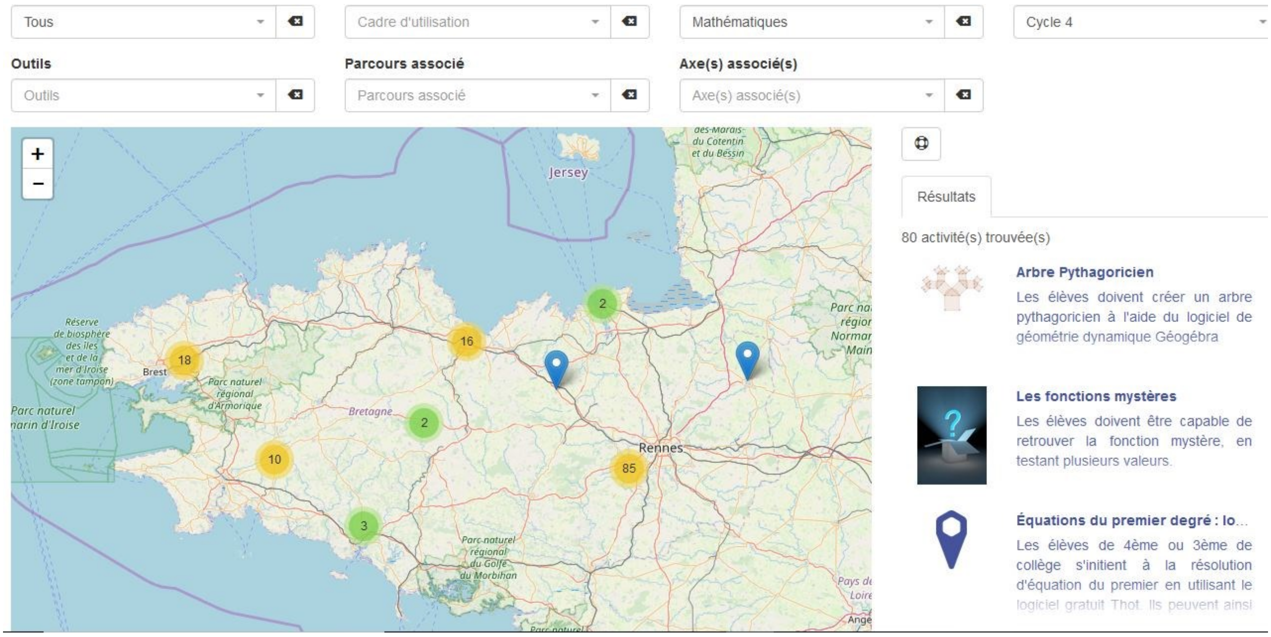

Figure 1. The CARTOUN Platform.

The platform has been designed with a clear instrumentation intention, to foster the use of digital tools in class. In terms of macro-level connectivity, the CARTOUN platform is clearly designed to create connections between authors and users. The content offered can be downloaded, and inserted in the teachers' resources system. On the opposite, any teacher can upload a lesson plan, but this requires filling different fields, it is not an immediate process, so the connections between the platform's content and the user's resources system are constrained. In the lesson plans, many external links are offered: to websites proposing to download software, or presenting a more detailed version of the lesson.

On the micro-level, I have chosen the theme of "First degree equations". Two lesson plans correspond to this theme. The first one focuses on solving such equations; students have access to computers and can use a software called "Thot" to help them. They have a list of equations to solve and they can work at their own pace. The second one is an activity in geometry (with Geoboard, a tool to construct figures on grids) using equations. I observe in these resources connections with software; between different kinds of representations; with the official competencies of the official curriculum, and for the second resource connections between equations and geometry, and between different possible solutions.

The Digital Resources for School Database

The DRSB platform for mathematics at cycle 4 is called "BAREM " (meaning Bank of Resources for Mathematics). The platform offers resources of different kinds. Some resources are called "bricks" (3056 bricks are available); others are called "modules" (394 modules), and associate several bricks. Twenty kinds of bricks are offered, for example: exercise, ICT exercise (involving software), interactive exercise, differentiated exercise, mind map, video, starter (short activity to start a lesson), teacher's sheet etc. The teacher can find resources on BAREM using several criteria: level, kind of resource, competencies, difficulty, theme, plus any kind of freely chosen keyword. Figure 2 below displays a BAREM screen, after the choice of the keyword "equations". The resources can be proposed to students working on the platform, or downloaded by the teacher; most of them can be modified. On BAREM the teacher can also upload his/her own resources, and can create his/her own modules or "paths" associating several bricks. The teacher can share resources with colleagues from his/her school and with other colleagues.

Figure 2. The Platform BAREM for mathematics in France.

The BAREM platform has clearly been designed to support the implementation of the new curriculum; we interpret this as an intention (of the ministry of Education) of fostering instrumentation processes of the users. The content corresponds to the new curriculum; moreover, several kinds of resources or tools correspond to policy priorities. The resources called "ICT exercises" support the use of GeoGebra and of the spreadsheet. The resources called "differentiated exercises" propose three versions of the same problem text, more or less guided. One of the search tools concerns the official competencies. At the same time the platform offers many possibilities for teacher design, including the possibility to modify some of the resources proposed, to upload his/her own resources, to share resources with colleagues. We interpret this as fostering instrumentalisation processes.

The concept of macro-level connectivity can enlighten these affordances of the platform. BAREM offers possibilities of connections: with the user's resource system, by downloading the platform's resources, but also by uploading his/her own resources on the platform; between different teachers by sharing resources; between the teacher and his/her students. Nevertheless, there is no possibility to contact the authors in order to suggest modifications. Moreover BAREM is only linked with the other platforms from the same publisher (concerning sciences, and German for cycle 4), but there are no external links.

Concerning the theme of “First degree equations”, 36 resources are available on BAREM. 28 of these resources are technical exercises: transforming and solving an equation. 2 exercises are in a geometric context, using a dynamic geometry software; 2 exercises a related to real-life situations; the remaining resources are 2 interactive exercises and 2 mind maps (concerning different kinds of equations). Thus at the micro-level for the theme, I observe connections between concepts (with the mind maps); connections with real-life situations and with geometry (but a reduced amount of those); connections with dynamic exercises.

DISCUSSION: COMPARING THE TWO PLATFORMS

The two platforms analysed here are both official platforms, and I claim that both of them embed institutional intentions that can be interpreted in terms of instrumentation processes. BAREM offers a content corresponding to the new official curriculum. CARTOUN is a platform for sharing lessons; nevertheless, these lessons follow a model proposed by the institution. Moreover some “official” groups (involving inspectors) share their work on CARTOUN. Both platforms are designed with an aim of evolutions of teachers’ practices: to foster the use of ICT; to support the implementation of the new curriculum, presented in terms of competencies. They are also designed with an intention of fostering teacher agency in different ways, and this appears in the analysis of their macro-level connectivity.

The macro-level connectivity of CARTOUN is especially developed in terms of possibilities of connections between teachers – which can be interpreted here also as connections between teachers and authors. BAREM offers the possibility of connections between teachers, if a teacher wants to share his/her lessons with colleagues. But there are no possible connections between teachers and authors. On the opposite, BAREM offers the possibility of connections between teachers and students, which does not exist on CARTOUN. Concerning connections with teachers’ resources systems, they are more developed on BAREM: a teacher can design a lesson, mixing in it BAREM resources and his/her own resources. On CARTOUN the teachers can download a lesson; but uploading is framed by different fields to be filled. About connections with external websites, they are more developed on CARTOUN since the authors are free to propose external links in their lesson plans.

In terms of micro-level connectivity, concerning equations, both platforms connect different representations, and connect equations and geometry. BAREM offers better connections between concepts with the mind map on equations; while CARTOUN offers better possibilities for addressing different students’ needs, with a lesson where students can solve equations at their own pace (and no “differentiated exercise” on this topic in BAREM). CARTOUN has also better connections with software, through external links inserted in the lessons.

CONCLUSION

The use of textbooks to support instructional reforms has been extensively studied (e.g. Ball & Cohen 1996). Digital platforms offer new means for this aim, and can lead to new phenomena. We claim that the documentational approach, and the concept of connectivity in particular, can enlighten these phenomena. In this study we illustrated it by the analysis and comparison of two platforms in France.

A study of the actual use of the platforms is necessary to complement this first analysis, from potential connectivity to actual connectivity in classroom uses. Since their use is not compulsory, and probably also because of the popularity of a previous platform: LaboMEP, developed by the Sésamath association (Gueudet *et al.*, 2018), finding mathematics teachers actually using these platforms is a complex task.

Another direction for a better understanding of the transformations of teachers' work produced by digital platforms is provided by the international comparison. This work was initiated in (Gueudet *et al.* to appear); this symposium is a new step in this comparative work.

References

Ball, D. L., & Cohen, D. (1996). Reform by the book: what is – or might be – the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6-8, 14.

Gueudet, G., Bueno-Ravel, L., Modeste, S., & Trouche, L. (2017). Curriculum in France: A national frame in transition. In D. Thompson, M. A. Huntley, & C. Suurtamm (Eds.), *International perspectives on mathematics curriculum* (pp. 41–70), Charlotte: International Age Publishing.

Gueudet, G., Pepin, B., Courtney, S., Kock, Z.-J., Misfeldt, M., & Tamborg, A. (to appear). Digital platforms for teacher documentation work: Affordances and constraints for teacher curriculum design. In A. Clark-Wilson et al. (Eds.), *Mathematics education in the digital age: learning, practice and theory.* London: Routledge.

Gueudet, G., Pepin, B., Sabra, H., Restrepo, A., & Trouche, L. (2018). E-textbooks and connectivity: Proposing an analytical framework. *International Journal for Science and Mathematics Education*, *16*(3), 539–558.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.). (2012). *From textbooks to 'lived' resources: Mathematics curriculum materials and teacher development*. New York: Springer.

Trouche, L., Gueudet, G., & Pepin, B. (Eds.). (2019). *The 'resource' approach to mathematics education*. Cham: Springer.

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Dordrecht: Kluwer.

PLATFORMS AS INFRASTRUCTURES FOR TEACHERS' WORK WITH RESOURCES

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In this paper, we describe how digital learning platforms can be analysed as "resources" using the documentational approach to didactics (DAD). We explore the case of Denmark where national platforms that are compulsory for mathematics teachers to use. These platforms are received both as useful by some teachers and as very problematic by others. We suggest that the compulsory nature of the implementation as well as the larger context of conflict between teachers and the government are critical factors explaining how the same technology are received so differently and we discuss how such phenomena concerning compulsory deep infrastructures can be seen with DAD.

INTRODUCTION

In many countries, teachers have access to digital platforms on which they can access content, share resources with other teachers and/or design their lessons. The design, affordances and constraints of these platforms may therefore be a significant factor for teachers' work. One of the variables that determine the design of platforms is their relation to policy, educational reforms and other systemic aspects at the systemic level of educational system. This is indeed the case of the digital platforms, which recently have been implemented in Danish compulsory schools. Unlike other types of digital platform, the Danish learning platforms do not provide teachers or students content (teaching materials or other similar pedagogical resources), but only the infrastructure for teachers to develop content, which they then can upload and share via the platform. The platforms are required to have a "forløbsbygger" (teaching sequence editor), which are designed to work as a template that can scaffold teachers in their planning of lessons and courses within the learning platform.

In correspondence with this symposium, this paper seeks to investigate how documentational approach to didactics (DAD) can contribute in investigating the constraints and affordances of the Danish digital usage in the context of their usage by mathematics teachers. To investigate this matter, we will present two examples that display affordances and constraints of the platforms when used by mathematics teachers. The first is a previously published example of three teachers' joint planning of a lesson in geometry in middle school (students aged 10–11), which show the platforms may support teachers in making decisions regarding the design of their lessons. The second example demonstrates how other teachers experience the very same structure that support the teachers in example 1 as highly constraining and rigid. Based on these examples, we argue that the outcomes and effects of using the

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 105–110). Paderborn: Universitätsbibliothek Paderborn.

platforms vary depending on the resource system and goals of the teacher using it. Both of the examples contributes in pursuing a purpose of investigating how and to what extent the documentational approach to didactics (DAD) can describe the occurring phenomena.

THE DANISH PLATFORMS

In 2014, the Danish government decided on a digitalization strategy for the Danish public sector, which included that all municipalities in Denmark should purchase and implement a digital learning platform before the end of 2017 (Misfeldt et al., 2018). Instead of centrally developing a national learning platform, the Ministry of Education and Local Government Denmark decided to develop 64 functional requirements for the platforms and leave it up to private manufacturers to produce them (Misfeldt et al., 2018). The responsibility of choosing, purchasing and implementing a platform was then left to the individual municipalities. Currently, there is 5 platforms available for municipalities to choose from. In this paper, we focus on one of the most widely used platforms called "Meebook". Among other things, the 64 requirements specified that the platforms should have a teacher interface that would provide teachers a digital infrastructure to design teaching sequences (add content, learning objectives, PDF document, links, photos, digital textbook material and other resources), to access digital textbooks from publishers, to evaluate students performances and allow teachers to distribute teaching content to their students. The platforms were therefore also required to include a student interface, in which students could access lesson plans and the resources included therein. Another key aspect of the requirements was that the platforms should integrate and support the implementation of a new curriculum launched in 2016. Contrary to the previous curriculum standards, this new curriculum focused on learning objectives organized in competence areas, skills and knowledge. In addition, the new curriculum and its learning objectives introduced a workflow where teachers were expected to planning lessons by selecting a learning objective from the national curriculum standards, then interpret it and 'break it down' into a more concrete objective for at specific lesson. These learning objectives should then function as the outset for the teachers' design of the lesson. The Danish learning platforms was to support this workflow by enabling teachers to access learning objectives from the platforms, allowing teachers to interpret them and to define content and resources in accordance with the learning objectives for the given lessons.

Figure 1 below shows Meebook's teaching sequence editor.

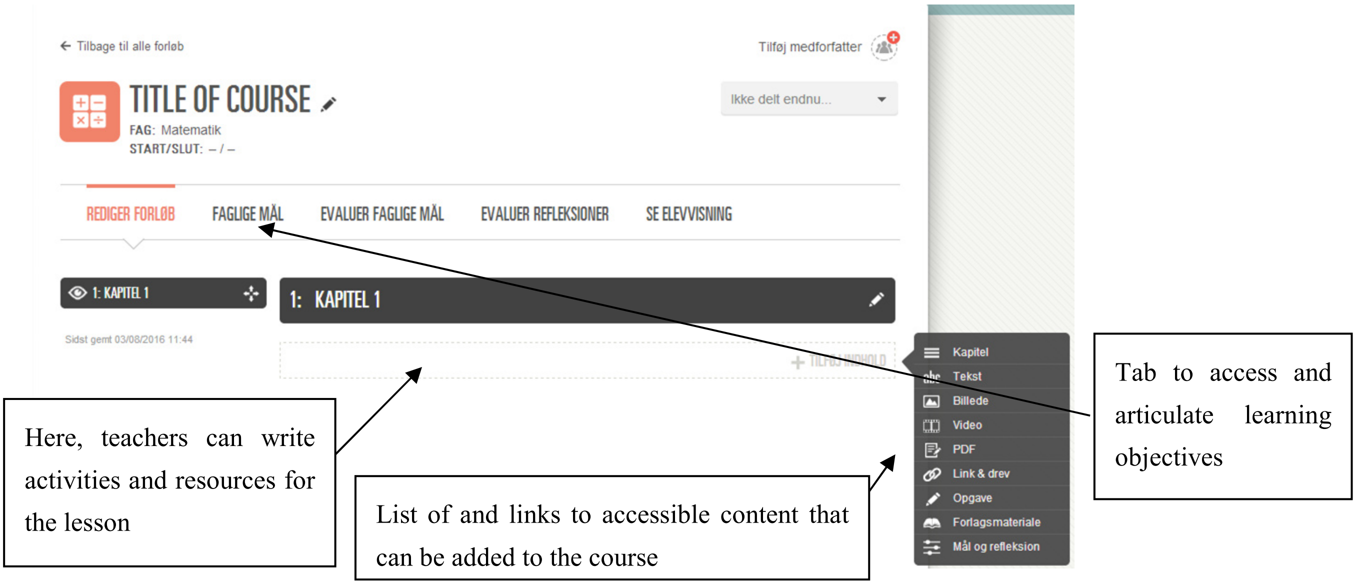

Figure 1. Meebook's interface for teachers to plan a course/lesson (the teacher can add a chapter, text, a picture, video material, a PDF document, a hyperlink, a task or activity, e-textbook material or a student reflection)

In the lesson sequence editor, teachers are required to write the activities of the lesson and to add any content needed. In this tab, teachers can also link to digital textbooks insofar that the school have purchased a subscription. After having defined activities and content for a lesson, teachers can move on to the interface represented in figure 2 below.

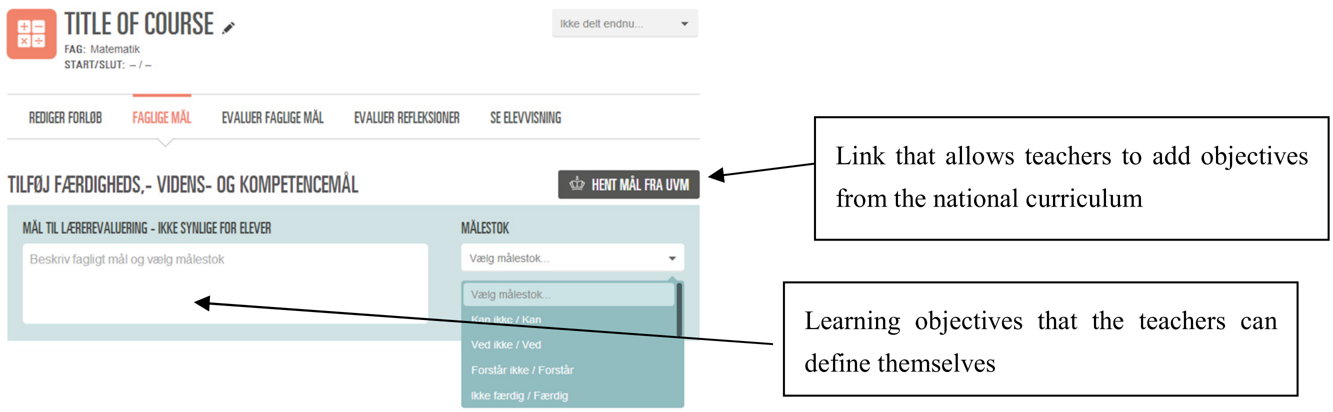

Figure 2. Meebook's interface in the tab called "Add skill, knowledge and competence objectives"

THEORETICAL APPROACH AND RESEARCH QUESTION

In the terminology of the DAD, mathematics teachers is believed to look for *resources* (Adler, 2000), choose resources (textbooks, digital tools, student production etc.), modify them and use them in class. While appropriating and using such resources, teachers develop a scheme of use, which, together with the resource, constitute a *document* (Pepin, Gueudet, Yerushalmy, Trouche, & Chazan, 2016). The features of the resources influence this development, in an instrumentation process; at the same time, teachers modify the resources they use according to their professional knowledge in an instrumentalisation process. The concepts introduced in DAD thereby provide a vocabulary to describe and analyze teachers' usage of platforms can modify or affect their work. A central characteristic of the Danish digital platforms is however that they are mandatory for teachers to use and that they, in addition, are built to 'encourage' a particular workflow for teachers preparation of lesson. This is a feature of the Danish platforms that seemingly is different from the assumption in DAD that teachers, largely at their own initiatives, are believed to look for and choose resources and develop schemes of usage. In the Danish context, the platform is a mandatory resource for teachers to use and it seeks to specify teachers' usage of other resources by introducing a template with a fixed workflow. In this paper, we therefore seeks to investigate the following research question:

How and to what extent can DAD contribute in describing and analyzing mathematics teachers' work with mandatory digital platforms?

To address this research question, we present two examples of mathematics teachers' usages of the platforms. The first example stems from a previously published case study (Tamborg, 2018) and shows how Meebook support three teachers' collaborate planning of a geometry lesson. The second example stems from an interview of a mathematics teacher conducted during a PhD study (Tamborg, 2018) and shows how Meebook's teaching sequencing editor is conceived as rigid and constraining. In each of these examples, we discuss how and to what extent DAD can study the occurring phenomena.

RESULTS

Affordances - collaborate planning with Meebook

The three teachers are planning a geometry lesson and alternately discuss the lesson and write their decisions into Meebook in the tab illustrated in figure 1 above. They decide that the students should categorize the geometric figures they have been working with lately and they decide that the students should work in groups. The teachers end up discussing whether the students should categorize the figures 'freely' or whether they should follow certain instructions. As the teachers discuss this matter without immediately reaching an agreement, one of the teachers turns to the tab in Meebook where they have written the learning objectives for the course and reads the objectives aloud to her colleagues: *"According to the objectives, the students should be able to distinguish between the five figures and categorize different types of figures according to their side lengths and angle sizes"*. Another teacher then argues that if these objectives should be addressed, then the students should identify the figures from their properties and that they therefore should be given instructions to do so. The two other teachers concede. They then go on to discuss how they can make sure that the students actually talk about the properties of the figure, and not just categorize them from what they believe the figures look like. Hence, the teachers find is likely that this method of categorizing the figures would not target the objective for the lesson. The teachers hinder this from happening by cutting the cardboard figures into shapes that are unlike the figures the students have been exposed to during the last two weeks. In the case above, by benchmarking their anticipation of how the students would engage in the a lesson design against the learning objectives written in Meebook, the teachers are able to make qualified decisions about both the choice and modification of resources and how they should be used (scheme of utilization). In this case, DAD thus provides important granular concepts to investigate the role of the platform in the three teachers' documentation work.

Constraints – limitations of a narrow template

Danish teachers however have diverse perspectives on the usefulness of the platforms, and several teachers even describe the platforms as counterproductive and unaligned with their pedagogical core values. This is illustrated by the following quote, which stems from an interview with a Danish mathematics teacher. This teacher, who also uses Meebook, explains how the learning objectives in Meebook fits poorly with his mathematics teaching:

"The entire pedagogical frame in the learning platform has a way too narrow focus on learning on learning objectives. One of the main ideas behind the platform is to focus on learning objectives instead of the content. For me, however, the content is by far the most important part of the lesson and the content that keeps the students motivated. Besides, I don't believe that we are able to anticipate what students will learning and to define the ahead of the lessons in a platform. The way I see it, the platforms needs to do something entirely different than what is the case now".

As shown in figure 2, Meebook interface requires teachers to 1) define a learning objective and 2) to choose one of four of Meebook's predefined measurements scales, which are to be used by teachers in the assessment of students after the lessons have been carried out. In the quote above, the teacher explains why this structure of the platforms is problematic from his point of view; it requires him to be able to anticipate exactly how students will interact with the content he has planned. In addition, mathematics teachers have argued that the binary measurement scales only applies to some parts of mathematics teaching. While it might make sense to assess whether or not students are capable of adding two fractions, such concrete learning objectives are not useful when it comes to training students' spatial intelligence. Within DAD, this case can be analyzed a situation where the teacher experience Meebook's teaching sequence editor to enforce a rigid scheme of utilization, which determines his usage of the resources integrated in the platform. The characteristic of the platform that causes this experience however stems from the wider political context surrounding the platforms – namely that they are developed to support the implementation of a curriculum reform and that they are mandatory to use. This is thus not an issue that adequately can be analyzed only as a result of micro cognitive processes.

DISCUSSION AND CONCLUSION

By some teachers, the Danish platforms are conceived of having highly rigid scheme-like structures that force teachers into pre-defined and narrow workflows, while other believe the platforms to create a well-functioning foundation for collaboration on how to design lessons. It is not at all difficult to understand that different people's usage leads to different outcomes in terms of how satisfied teachers are with the technology. In DAD, it is indeed a relatively trivial point that different teachers' usage of the same resource leads to different outcomes. The cases described and analysed above shows that DAD to a wide degree provides appropriate and useful concepts to study how this is the case. DAD can allow us to understand exactly *how* platforms modify teachers' documentation work and provides reasonable insights about *why* some teachers might dislike or even recent the platforms. The cases presented above moreover shows that the potentials of platforms are dependent on that teachers comply with the ideas and approach of the platforms, which is not always the case. The reasons for non-compliance cannot be found at the cognitive micro level, but regard affective, organizational and political issues, which we have described elsewhere (Misfeldt et al., 2018). As witnessed by the contributions of this symposium, there are substantial differences from country to country in how platforms are connected to the educational system; are the platforms developed centrally from state-driven initiatives or by private manufacturers? For what purposes have the platforms been developed? Are they mandatory of voluntary to use? The answers to these questions all have a significance for mathematics teachers' work with them, which cannot necessarily be observed or fully comprehended from observations at the micro level. It is therefore our hope that this paper can spark a discussion of how we can maintain the value of DAD when studying such processes while simultaneously being able to account for contextual factors at the meso and macro level.

References

Adler, J. (1998). Resources as a verb: Recontextualising resources in and for school mathematics practice. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 1, pp. 1–18). Stellenbosch: University of Stellenbosch.

Misfeldt, M., Tamborg, A. L., Qvortrup, A., Petersen, C. K., Svensson, L. Ø., Allsopp, B. B., & Dirckinck-Holmfeld, L. (2018). Implementering af læringsplatforme: Brug, værdier og samarbejde. *Læring og Medier*, *10*(18). doi.org/10.7146/ lom.v10i18.97013

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. I. (2016). E-Textbooks in/for teaching and learning mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education, third edition* (pp. 636–661). New York: Routledge.

Tamborg, A. L. (2017). Planning geometry lessons with learning platforms. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10) (*pp. 2430–2437). Institute of Education, Dublin City University, Ireland, and ERME.

FROM PLATFORM ACCESS TO TEACHER DESIGN WORK: CONSTRAINTS AND AFFORDANCES OF PLATFORMS — THE DUTCH CASE OF WIKIWIJS

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In this paper we analyse a Dutch digital platform regarding its affordances (and constraints) for teachers' documentation work in terms of "connectivity" at macro (external connections) and micro (internal connections) level. Results showed that at macro level the platform supported the import, modification and use of open shareable resource, as well as teacher collaborative activities with/on educational resources. Moreover, it allowed to some extent for assessment and differentiation to be linked to the designed resources, but not for teacher-teacher, or teacher-student communication. At micro level the functionality allowed connections to be made "internally", for example between a topic and previous or further knowledge; albeit this functionality was seldom used and no interactive features could be found.

INTRODUCTION

Worldwide, digital platforms have become available to support teachers, working individually or collaboratively, in their planning, preparing, and enacting various aspects of their work. These platforms offer new opportunities for mathematics teachers to use digital technology in order to prepare and foster student learning. However, platforms may also pose constraints on teachers' work, for example for technical reasons or as a result of decisions made by national authorities who initiated the platforms. This paper is part of a seminar describing and comparing digital platforms, in Denmark, France, the USA, and the Netherland. In this paper we describe and analyse the case of a government funded educational platform in the Netherlands ("Wikiwijs"), leaning on the theoretical frame of the Documentational Approach to Didactics (DAD, Trouche, Gueudet, & Pepin, 2019), and the notion of *connectivity* (Gueudet, Pepin, Restrepo, Sabra, & Trouche, 2018). We address the following research question: *In terms of connectivity, what are the affordances and constraints for mathematics teachers offered by the Wikiwijs platform*? In the following sections we describe, first, the context of the platform; second, the theoretical frame; third, the method/s used to analyse the platform; fourth, the results from this analysis; and fifth, conclusions.

CONTEXT

In the Netherlands schools and teachers are free to create/design their own set of educational resources, be they digital or traditional, freely provided or commercially produced, as long as they teach the content according to the National Curriculum

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 111–116). Paderborn: Universitätsbibliothek Paderborn.

framework. In 2008 the Dutch government advisory board for education ("Onderwijsraad") published a report, in which it recommended to encourage the use of Open Educational Resources (OER), and supporting teachers to "arrange and develop" digital resources (Onderwijsraad, 2008). In response to these recommendations, a freely accessible platform "Wikiwijs" (www.wikiwijs.nl), was launched in 2009 by the government funded ICT service provider to the field of education ("Kennisnet"). On this platform teachers (at any education level) can find, use, adapt, design and share digital educational resources. It is expected that Wikiwijs will draw approximately 9 million separate visits in 2019 (Kennisnet, 2019). However, in an online survey among mathematics teachers (Kock & Pepin, 2019) the use of Wikiwijs scored relatively low, compared to other digital resources (e.g. digital versions of mathematics textbooks; textbook companion sites).

THEORETICAL FRAME

We summarize the main concepts of the theoretical frame and refer to (Gueudet et al., 2019) for a more comprehensive description.

Following the DAD perspective, we contend that in their daily work teachers search for, select, modify and arrange resources. This activity is called teacher documentation work (Trouche, Gueudet, & Pepin, 2018) and in this process teachers produce "documents": these are the resources combined with their schemes of use (Vergnaud, 1998), which are defined as a coherent set of notions and rules related to working with the resources. The development process of a document is called documentational genesis. The affordances and constraints of the resource/s shape the schemes developed by a teacher; this is called the instrumentation process. Simultaneously, a teacher's knowledge and beliefs shape the way the teacher uses, appropriates and transforms the resources; this is called the instrumentalization process. Social and institutional factors influence teachers' documentation work.

Teachers' documentation work also takes place when they access and work with (the resources on) digital platforms. These platforms provide particular tools to support teacher design activities, an educational focus, and a structure to encourage particular ways of working, envisaged by authorities who initiated the platform. Thus, by design the platforms encourage specific instrumentation and instrumentalization processes.

We analyse the affordances and constraints of a platform (in terms of opportunities for learning) using the concept of *connectivity*, originally devised to analyse e-textbooks (Pepin, Gueudet, Yerushalmy, & Chazan, 2016). A further distinction is made between macro-level connectivity (referring to connections between the platform and other resources, or between users and their resources systems), and micro-level connectivity (referring to connections within the mathematical domain, between representations, or between mathematics and other subjects).

METHOD

First, we have described the general structure of the platform by examining its home page and the pages pertaining to its two functions, consulting the platform's documentation if necessary. Second, we have examined examples of Wikiwijs content related to mathematics.

To analyse the affordances and constraints of Wikiwijs in terms of macro- and micro-level connectivity, we used the analysis grid developed by Gueudet et al. (2018). To analyse the macro-level connectivity, we noted for each connectivity item of the grid, if it could be realized by the Wikiwijs functionality. Moreover, we looked for example in the Wikiwijs content of such realisation, first within the mathematics content, and if this could not be found, in the content of other subjects.

To analyse the micro-level connectivity, we applied the analysis grid to a particular mathematics example. In April 2019 a Wikiwijs search for mathematics (”wiskunde”) in combination with ”Wikiwijs arrangement” gave 838 results. After filtering to find the Wikiwijs arrangements created by (groups of) teachers, we obtained 350 results. We examined a selection of the filtered Wikiwijs arrangements, by different authors, and noted that several consisted of empty pages or non-functional links to external content. For further analysis, we selected one learning arrangement, adapted by secondary school mathematic teachers from a VO-content Wikiwijs arrangement. This was a learning arrangement on ”similarity” (of triangles and other shapes) for grade 7-9. In the analysis we also referred to other Wikiwijs arrangements we examined, but a full analysis of all the Wikiwijs arrangements on mathematics was not possible in the scope of this paper.

DESCRIPTION OF THE PLATFORM STRUCTURE

Wikiwijs is accessed through a browser. Its main page shows (1) a ”search” function, and (2) a ”create/design” function. (1) The search function searches for educational resources in a set of educational databases and repositories, including Wikiwijs itself. It uses metadata to enable filtering and meaningful searches, such as the educational level, examination syllabus learning aim, type of material (e.g. test, task, lesson or lesson series), subject content, source (e.g. YouTube, Wikiwijs, a publisher's repository), and quality labels to enable filtering and to make searches more meaningful (e.g. assigned by a group of teachers, the “learning material specialists”). See Figure 1 for an example of search results. (2) The “create/design” function offers the possibility to create/design new, or modify existing structured sets of educational resources called Wikiwijs ‘arrangements’. Typically, Wikiwijs arrangements contain a menu structure and content consisting of text in combination with (links to) pictures, audio/video, documents and links to websites. Authors may link to (parts of) other Wikiwijs arrangements, or include a modified version of these arrangements in their own. Additional functionality enables the creation of different types of questions or quizzes and tests with feedback. Each Wikiwijs arrangement ends with a colophon indicating, among others, the author(s) and a Creative Commons copyright notice.

Wikiwijs supports the definition of teams and allows teamwork on learning arrangements.

Teachers and students can access Wikiwijs arrangements through the search function or by URL, and can download them as documents (pdf, or eBook). Alternatively, teachers can link to, or import, Wikiwijs arrangements into the Digital Learning Environment (DLE) of their school. It is through integration into the DLE that the platform enables differentiation of content and storage of students' test results.

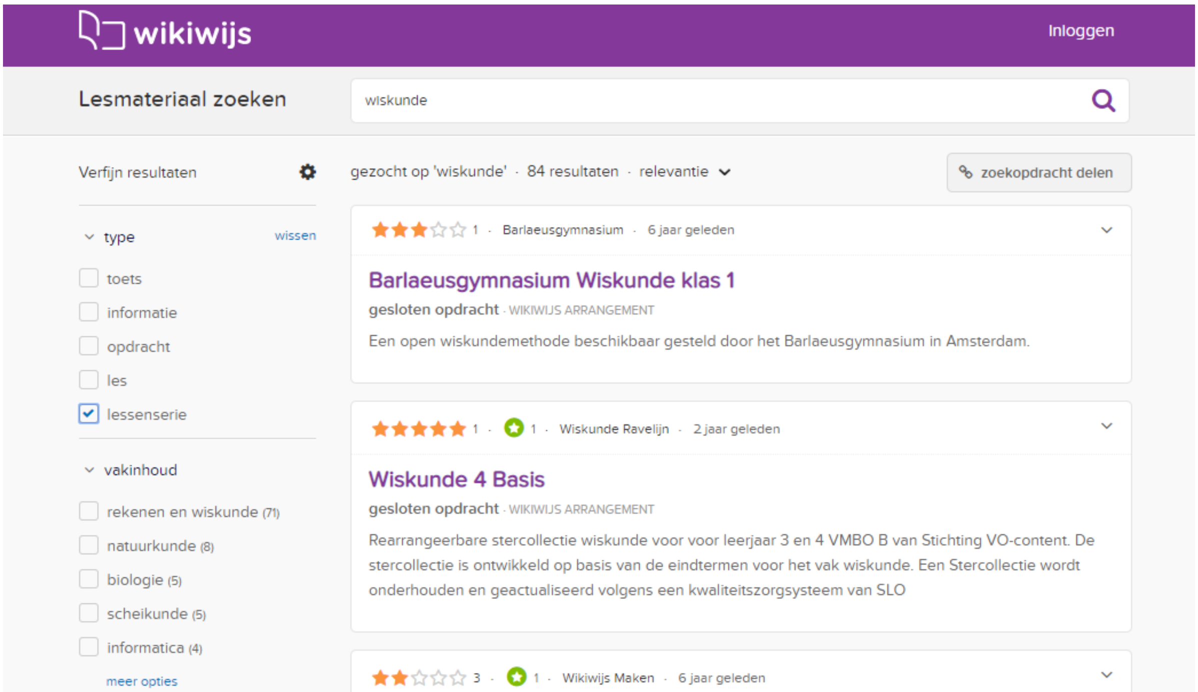

Figure 1. The Platform Wikiwijs in the Netherlands (search results)

At this moment one not-for-profit organization, VO-content (see www.vo-content.nl), has published its learning materials on the platform and offers additional products and services (e.g. textbook on paper, teacher training) at a fee to subscribing schools.

WIKIWIJS AT MACRO-LEVEL

We contend that the Wikiwijs platform provides the following affordances in terms of macro-level connectivity: (1) Provision of open educational resources, searchable using relevant metadata; downloads of Wikiwijs arrangements (pdf or e-book) and integration into DLEs. (2) Individual or collaborative rearrangement and modification, as well as creation of Wikiwijs arrangements; evaluation / certification of content by expert teachers. (3) Links to resources external as well as internal to Wikiwijs. (4) Links to the national curriculum can be made by means of meta-data. (5) Exercises and (formative) tests with feedback. Macro-level constraints are that Wikiwijs offers no area for teacher-teacher or teacher-student communication, and no data storage (e.g. test results) at a student level.

WIKIWIJS AT MICRO-LEVEL

At the micro-level we observed, based on the analysis of the "Wikiwijs maken" functionality and the illustrative example on similarity: (1) The platform allowed connections between a topic and previous or further knowledge, different concepts or different topic areas and disciplines. However, in the Wikiwijs arrangements we examined only limited connections to previous knowledge; everyday situations/contexts were present. (2) In order to connect to different students' needs, some learning arrangements used colour-coded tasks for high achieving students. In general, the standard menu structure of the Wikiwijs arrangements seemed to favour a linear path through the content. (3) Connections to different mathematical representations (e.g. geometric and algebraic representations) and to different moments of appropriation were made: content explanations used text, diagrams, formulas and (links to) explanatory videos. However, dynamic mathematical content (e.g. the manipulation of mathematical objects in graphs; repeat exercises with different numbers) was not supported. (4) The Wikiwijs arrangement on 'similarity' connected to exercises and assessment in the form of a link to an assessment rubric (external file) and paper and pencil student exercises, with "check your answer" buttons.

DISCUSSION AND CONCLUSION

In terms of connectivity, and at macro-level, we found that the platform supports the creation, modification and use of open shareable resources which can be linked to resources inside and outside of the platform. The platform also supports collaborative activities of teachers on educational resources. The possibility to attach educationally relevant meta-data to digital resources enhances the search functionality, and hence makes resources from various repositories and databases more accessible for teachers. Wikiwijs offers no area for teacher-teacher or teacher-student communication, or data storage (e.g. test results) at a student level, which can be considered macro-level constraints. The examples we studied indicate that at the macro-level the platform supports instrumentation processes, all in line with the aims of the initiators of the platform. This has the potential to transform teachers' documentation work (Pepin et al., 2016).

At micro-level, that is the level of mathematical content, the functionality allows connections to be made between a topic and previous or further knowledge, different concepts or different topic areas and disciplines. However, when analysing the use of the platform, this has been used only to a limited extent. The standard menu structure of the learning arrangements seems to favour a linear path through the content, although some learning arrangements used particular tasks for differentiation (e.g. high achieving students). The example we studied contained different representations of mathematical content, but dynamic content, such as manipulation of mathematical objects in graphs or redoing exercises with different numbers, was not supported.

It appears that at the micro-level, the affordances of the platform in terms of connectivity (e.g. to different topics, concepts, and grade levels; or to different contexts and subjects) have not been fully exploited. Such opportunities are possible, but not inherent in the structure of the platform. Their realization depends on how the teachers, based on their beliefs about mathematics learning, create these opportunities by means of the platform (the instrumentalisation process). In this regard, the lack of functionality to support specific mathematical learning opportunities (such as dynamic mathematical content) can be considered a real constraint.

This lack of specific mathematical learning opportunities may be one of the reasons for the limited use mathematics teachers make of Wikiwijs (Kock & Pepin, 2019). However, practical (e.g. time to collaborate) or institutional reasons (e.g. support from the school administration) may also play a role (Gueudet et al., 2019), and hence the use of platforms such as Wikiwijs is a crucial issue for further research.

References

Gueudet, G., Courtney, S., Kock, Z.-J., Pepin, B., Misfeldt, M., & Tamborg, A. (2019). Digital platforms for teacher documentation work: Affordances and constraints for teacher curriculum design [Manuscript in preparation].

Gueudet, G., Pepin, B., Restrepo, A., Sabra, H., & Trouche, L. (2018). E-textbooks and connectivity: Proposing an analytical framework. *International Journal of Science and Mathematics Education, 16*(3), 539–558. doi:10.1007/s10763-016-9782-2

Kennisnet (2019). *Jaarplan 2019 [Annual plan 2019].* Retrieved from https://www.kennisnet.nl/fileadmin/kennisnet/publicatie/Jaarplan_2019.pdf

Kock, Z.-J., & Pepin, B. (2019). Secondary school mathematics teachers' selection and use of resources. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Onderwijsraad. (2008). *Onderwijs en open leermiddelen [Education and open educational materials]*. The Hague, the Netherlands: de Onderwijsraad.

Pepin, B., Gueudet, G., Yerushalmy, M., & Chazan, D. (2016). E-textbooks in/for teaching and learning mathematics: A disruptive and potentially transformative educational technology. In L. D. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (pp. 636–661). London: Taylor & Franics.

Trouche, L., Gueudet, G., & Pepin, B. (2018). Documentational Approach to Didactics. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 1–11). Cham: Springer.

Trouche, L., Gueudet, G., & Pepin, B. (Eds.). (2019). *The 'resource' approach to mathematics education*. Cham: Springer.

Vergnaud. (1998). Towards a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity.* (pp. 227–241). Dordrecht: Kluwer.

Part III.

Workshops

PRETEXT AUTHORING WORKSHOP

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PreTeXt is an authoring and publishing system, used primarily (but not exclusively) to create and distribute undergraduate mathematics textbooks. This workshop will help participants understand how careful source mark-up by authors can dramatically increase the ability to easily collect data about textbook use. The workshop will be very hands-on, and participants will complete a short book which they will convert to both online and print formats.

PRETEXT

PreTeXt (https://pretextbook.org) is an authoring and publishing system, used primarily (but not exclusively) to create and distribute undergraduate mathematics textbooks. There are presently about sixty textbooks authored with PreTeXt, mostly published with open licenses. A key part of the design is that authors create source material in a very structured form. This allows us to replicate that structure within the electronic versions produced—in ways invisible to the reader, but such that it is possible to very accurately observe how a reader interacts with their book. (See (O'Halloran, Beezer, & Farmer, 2017) for more details.

AUTHORING

During this hands-on workshop, participants will create a short book in PreTeXt, which will amply demonstrate the system. We will then explain the implications for collecting data on readers' use of textbooks.

Participants will need to bring their Internet-connected laptop equipped with a recent Chrome or Firefox web browser. We will work online in CoCalc (https://cocalc.com), so you can prepare by making a free account there in advance. Note the email address you use when you sign up, and email it to beezer@ups.edu with the subject line "ICMT3 Workshop". We can also setup CoCalc just prior to the workshop.

References

O'Halloran, K. L., Beezer, R. A., & Farmer, D. W. (2018). A new generation of mathematics textbook research and development. *ZDM Mathematics Education, 50*(7), 863–879.

THE DEVELOPMENT OF PROJECT-BASED LEARNING TEXTBOOK IN MATHEMATICS: THEORY AND PRACTICE

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In the last decade, project-based learning (PBL) has been increasingly applied to discipline education. However, it was not the same in mathematics education, especially in mathematics curriculum and textbook development. Therefore, our group has kept conducting theoretical and practical research on PBL application. In a five-year project, our research group expects to stimulate students' interests in mathematics and improve the quality of mathematics learning through the development of PBL mathematics textbook (PBL-MT). We will discuss following research questions in this workshop:

Q1: How does PBL affect students' learning?

Based on literature review, we find PBL has the following advantages:

Firstly, PBL has positive effects on students’ non-intellectual factors, such as motivation (Verma, Dickerson, & McKinney, 2011), metacognition (Sart, 2014) and self-efficacy (Chen, 2015).

Secondly, PBL has positive influences on low learning achievement students (Halvorsen et al., 2012) and students with learning disability (Filippatou & Kaldi, 2010).

Although research about student learning are common in other disciplines, there is relatively few empirical researches worldwide focusing on PBL’s application in mathematics and the influence of its application (Hudson, 2010), let alone in China.

Q2: How to compile middle school’s mathematics textbooks based on PBL? And what are the effects?

In order to answer Q2, our research group will introduce our theoretical and practical research related to PBL textbook development and developed PBL-MT. Theoretically, based on PBL and mathematics curriculum theory, our team constructed a framework about PBL-MT. Practically, we explored the textbook development and instruction of PBL-MT. Firstly, we described functions, values, outlines, as well as the principles of designing PBL-MT, which provided a unified standard for the design. Secondly, based on the current mathematics curriculum standards and textbooks, we systematically developed over thirty PBL-MT units for middle school.

In PBL-MT development, the relationship between the knowledge structure, project activities, as well as the balance between contextualization and mathematics should be considered. Meanwhile, it is also necessary to deal with the contradiction between PBL

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 121–122). Paderborn: Universitätsbibliothek.

and standardized test, and the relationship between PBL and traditional instruction in mathematics.

Our team has prompted a series of research to apply PBL-MT at experiment schools. In the video session, the audience could find out how we apply PBL-MT in middle school and its possible effects.

In our instruction experiment, we concluded that the PBL-MT could meet middle school students' cognitive style and their learning requirements. Moreover, PBL-MT had significant positive impacts not only on the development of students' mathematical abilities in problem solving, problem exploring, and innovation, but also on some non-intellectual perspectives, such as their learning attitudes and interests in mathematics. Therefore, we believe that PBL-MT could be used as an auxiliary learning material. It could be used to intervene in gifted or low learners' mathematical learning process.

Moreover, in the video, our group member will briefly explain PBL-MT development; the teacher in our experiment school will discuss how to apply PBL-MT in his/her mathematics courses; also, the students who use PBL-MT are invited to talk about his/her learning experience.

During the workshop, the audiences could read our sample PBL-MT.

References

Alkan, F., & Emine Erdem. (2012). The relationship between teacher self-efficacy and competency receptions of chemistry teacher candidates. *Procedia – Social and Behavioral Sciences*, *47*, 1927–1932.

Chen, P., Hernandez, A., & Dong, J. (2015). Impact of collaborative project-based learning on self-efficacy of urban minority students in engineering. *Journal of Urban Learning Teaching & Research*, *11*.

Filippatou, D., & Kaldi, S. (2010). The effectiveness of project-based learning on pupils with learning difficulties regarding academic performance, group work and motivation. *International Journal of Special Education*, *25*(1), 17–26.

Halvorsen, A.-L. et al. (2012). Narrowing the achievement gap in second-grade social studies and content area literacy: the promise of a project-based approach. Working paper 26. *Theory & Research in Social Education*, *40*(3), 198–229.

Hudson, R. A. (2010). *Statistical argumentation in project-based learning environments.* Dissertations & Thesis – Gradworks, *256*.

Sart, G. (2014). The effects of the development of metacognition on project-based learning. *Social and Behavioral Sciences*, *152*(112), 131–136.

Verma, A. K., Dickerson, D., & Mckinney, S. (2011). Engaging students in stem careers with project-based learning-marinetech project: Marinetech addresses the urgent need to enhance underrepresented students' interest and performance in stem courses while fostering skills that are important prerequisites for. *Technology & Engineering Teacher*, *71*, 25–31.

Part IV.

Oral Communications

ON THE MARKETING AND DISTRIBUTION OF SCHOOL MATHEMATICS TEXTBOOKS IN A NEOLIBERAL CONTEXT

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This paper aims at troubling the distribution and market strategies of school mathematics textbooks in a context embedded in neoliberal policies and assumptions about education. And so, it explores the dynamic of the production, distribution, and selection of the official mathematics textbooks for Chilean schools and the market that unfolds simultaneously of non-official mathematics textbooks. Some of these non-official textbooks—meaning the Chilean Ministry of Education does not distribute these—are been sold at excessively high rates in bookstores. The paper grasps how circulating narratives, within neoliberal-based market strategies, have been governing the selection process of textbooks in some Chilean schools. This has led to some schools deciding not to use the official mathematics textbooks that are freely distributed by the Chilean Ministry of Education, and, rather, asking parents to buy the other expensive, and "assumed-as-better" mathematics textbooks for their children's education.

INTRODUCTION

This paper aims at problematizing, from a socio-political perspective, how neoliberal-based assumptions have travelled to govern most aspects of contemporary education. The unfolding of this paper sets an exploratory approach by following a rhizomatic (Deleuze & Guattari, 1987) analytical strategy. The rhizome enables to unfold a network of multiple dimensions that allows considering materials from the role of the media—in the mediatisation of education—as well as research that has argued on the market as part of the neoliberal structuring of education. In this light, the data gathered for this paper, taken as the first approximation to the research problem, corresponds to media reports that have addressed the production, distribution, and selection mathematics textbooks for Chilean schools and the market that unfolds simultaneously with it. It is an exploratory approach since the "economic spectrum" of education is not a usual object of study within research in the field of mathematics education. As Pais (2012, p. 70) contends "we lack research that explicitly connects these social phenomena—shamefully associated with school mathematics—with the broader political and economical spectrum. How, then, shall we understand the relation between school and capitalism?" And, as evidenced by the released of ICME 13th monograph "*Research on mathematics textbooks and teachers' resources: Advances and issues*" (Fan, Trouche, Qi, Rezat, & Visnovska, 2018), the socio-political perspective on mathematics education has not been widely explored.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 125–130). Paderborn: Universitätsbibliothek Paderborn.

The interest of looking at school mathematics textbooks, from this perspective, comes from the Chilean struggles to fine-tune educational policies towards providing equal opportunities to all students. And also, because school textbooks are either distributed without cost to public and subsidized schools and sell in bookstores to parents at high rates. Chile, acknowledged as Milton Friedman's—an American economist at the University of Chicago—"neoliberal experiment," embodies a rich scenario for understanding and troubling the marketing and distribution strategies behind the production and selection of mathematics textbooks. Neoliberal ideas in Chile, according to Miñana and Rodríguez (2003, p. 39), "began to spread widely since the mid-50s, but were relatively marginal and only went to guide government decisions in a systematic way thanks to the dictatorship governments in Chile, the place of the first national-level neoliberal experiment."

A BRIEF TAKE ON NEOLIBERALISM IN CHILE

Neoliberalism, in Chile, was unfolded as a form of securing education and economic stability (see Friedman, 1962). Within this mindset, schools were better managed if these were treated as private enterprises with a "consumer" focus (Chubb & Moe, 1990). Schools began to be embedded in a free-choice market in which every decision schools made aimed at securing their product: quality education, within which standardized tests constituted an "important resource" to shape the Chilean education system (Campos-Martínez, Corbalán, & Inzunza, 2015). Here, competition became a key component for the neoliberal school system, by delineating safe strategies—i.e. privatization—as an engine for establishing ways of improving schools. According to Cabalin, (2012, p. 233), this desire for improvement allowed to believe that "by introducing more privatization, schools will have to improve because they will have to compete for students, while also arguing that parents will have more freedom to choose the best school for their children due to this competition." This led to schools competing with each other at the national level through the mechanism called *System for Measuring the Quality of Education*—Sistema de Medición de la Calidad de la Educación (SIMCE). And so, schools are part of an assessment process in which every aspect of schools is submerged into public scrutiny.

SIMCE was a system designed for locating schools in need of management to improve the quality of education, however, over the years, it was taken as a mean "to inform parent-customers about the quality of service provision and to assist them in the selection of an educational establishment for their children" (Benveniste, 2002, p. 103). In this regard, the consumers of education are been informed about the status of their investments through SIMCE. Under a neoliberal mentality, "it is assumed that the public sector is dominated by inertia, that they have no motivation to better serve what is ultimately a captive clientele that cannot change providers, no matter how bad the service is" (Escalante, 2015, p. 206). Therefore, parents, as customers, read SIMCE reports and understand that the safest option is to pay for a school with the highest ranking they can afford, under a neoliberal mindset of 'if you pay more, you will have a better product." These constitute the first steps into a marketable school mathematics education (see Andrade-Molina, 2017), and, therefore, into a lucrative market of mathematics school textbooks as a mean of securing higher SIMCE scores.

ON THE FREE SCHOOL MATHEMATICS TEXTBOOKS

School textbooks in Chile are distributed, since 2000 (Olivera, 2017), by MINEDUC to all public and subsidized schools, through the *Assessment and Curriculum Unity*—Unidad de Currículum y Evaluación—: "in order to provide free coverage of school texts and, thereby, to promote an equitable and quality educational system that contributes to comprehensive education ensuring equal opportunities in education" (Gobierno de Chile, 2018, p. 271). The process of selection regarding which editorial will be the "official one" goes under a rigorous process that guarantees the closeness of each book under MINEDUC standards for each subject (see Gobierno de Chile, 2018). Each text is free of cost for every student in such schools, however, MINEDUC pays to the editorials the cost of manufacturing and distributing, subjected to a contract by both parts—MINEDUC and the editorial. (see Gobierno de Chile, 2018). As reported by *DiarioUchile* (Rojas, 2015), the national coordinator of curriculum and assessment asserts that MINEDUC spends approximately 36000USD each year in books for public and free distribution. Which means that the selection, production and distribution of textbooks become an investment that the Chilean government covers every year to secure the promised quality. MINEDUC also performs studies to evaluate the use and value of textbooks within the universe of schools that use these materials (see Olivera, 2017).

School textbooks distributed by MINEDUC are considered to be "a key tool in the teaching and learning process and a relevant curricular mean to progressively access [students] own skills, knowledge and attitudes of the subjects" (MINEDUC, n.d., par. 2). However, some schools argue on the restrictive matter of these free materials, namely that these do not provide tools for students with special needs as part of educational programs for inclusion (see Almazabar, 2018). Under this light, the public legitimization of school textbooks is aligned with each school project, something that, apparently, MINEDUC does not cover. Although, the problem of the market behind school textbooks is not recent and is prior to the law of inclusion—which dates from 2017 (MINEDUC, 2017)—, and, therefore, this is not the only argument for not using the textbooks. Moreover, these materials are taken as guidelines for schools; these are no restrictions on how to teach and learn school mathematics. The rejection of the use of MINEDUC's school textbooks has been reported in the media for several years. For González and Parra (2016, p. 85) "it is worth highlighting the case of school texts where an oligopoly has been constituted during these last 15 years. By 2015, three companies control the state funds allocated to this item."

ARE EDITORIALS FORCING SCHOOLS TO BUY THEIR PRDUCTS?

Besides good intentions of schools to search for the most fitted textbooks according to their own school projects, it has been reported by the media that the marketing strategies of non-official editorials have been somewhat non-academic. As Pais (2012, p. 70) asserts, "educational industries, from publishing houses producing textbooks to computer firms developing technology, see schools as a profitable market." However, research in the field of mathematics education has not yet addressed this issue. It is possible to see how socioeconomic gaps—in PISA or SIMCE—between students' achievement in mathematics have increased over the years, without approaching this phenomenon as a political and economical matter. As an example of the viewing of schools as a profitable market by editorials, Rojas (2015) reported in *DiarioUchile* the experience of a school principal with some school textbooks sellers:

> "As a school principal I received the text sellers, they were very interested in making an agreement, for which, in fact, they offered us even trips abroad to the management board and a lot of additional royalties, such as workshops for teachers, audiovisual teaching materials, book donations, […] We reviewed and compared Mineduc texts and they were practically the same. Finally, the school's advocate decided that we should ascribe to the purchased texts, rejecting the free ones by Mineduc, in spite of our report, arguing that, in this way, it is better filtered the type of student that they wished to capture, that is to say, camouflaged segregation was developed."

From this experience, it is visible a form of segregation that some schools might promote and is highly violent: students that cannot afford schools textbooks should not attend this school. Apparently, the "attracting schools to ask parents to buy expensive textbooks" is not only a lucrative business for editorials but also a refined apparatus for the segregation of students in schools.

Schools could eventually decide not to use MINEDUC's textbooks. But, according to the ordinance N°53 (MINEDUC, 2016), schools have to inform parents and MINEDUC in advance regarding their decision, given that the cost of paid textbooks is approximately 225 USD—almost half of Chile's minimum wage. Also, when comparing the prices of paid textbooks, there is a slight difference, in price, between paid mathematics textbooks and other school subjects—also visible in the number of additional materials, such as "exercise booklets," that are sold separately. A paid mathematics school textbook for the second level of high school costs approximately 63USD, whereas a textbook from the same level, from the same editorial but on language and literature is approximately 52USD.

All of the issues regarding the marketing of, amongst many other, school textbooks have been informed to the Superintendence of Education (see Carrasco-Aguilar, Ascorra, López, & Álvarez, 2018). Nevertheless, as Miñana and Rodríguez (2003, p. 18) assert "the capital and the market do not care if it sells bread, information, ecological tourism, automobiles, illusions or instruction; what is really important is the profitability of the investment, of what is sold and exchanged." They continue arguing that popularity ratings, such as the "best school", the "best teacher", and so on, —that have been defined by the outcomes of standardized tests and other indicators for assessing teachers and schools—have dislocated the "good intentions of policies" into mechanisms of competition and control. Here, the quality of textbooks become key in setting a mentality of quality equals the value of a product.

Following this train of thought, mathematics has been recognized as a particular school subject that leads to a competition of students, teachers, and schools. Andrade-Molina (2017) explores how, since the dictatorship in Chile, school mathematics became inserted within the national belief of shaping suitable citizens for the development of the nation. It is not of surprise, by the status given to mathematics,, that school mathematics textbooks, as well as other key subjects, become profitable products for editorial.

FURTHER DISCUSSION

It is clear that this is only a superficial glimpse on how neoliberal narratives have circulated in almost all aspect of the educational system in Chile. This paper aims at delineating the beginning of an exploration that needs to be done in the field of mathematics education. It is preoccupying to see how some aspects of mathematics education, such as the use of textbooks in the classroom, have become a place were marketing strategies have normed and governed schools' decision-making. And, despite MINEDUC's protocols for selecting which textbooks are going to be distributed, there exists the narrative that free textbooks are of lesser quality. Naturalizing such neoliberal ideas is dangerous, as aforementioned, for the segregation of socioeconomic disadvantaged students. As a further exploration, we will continue this inquiry with a case study of a school in Santiago, Chile, that has had a similar experience of accepting textbooks sellers' offers. The aim is to explore the perspective of mathematics teachers that have to deal with these decisions and to, at the same time, evaluate if these decisions contribute to increasing the socioeconomic gap in Chile. Unfortunately, we have no solution for this rather than to express our concerns on the fact that, probably, education in Chile has entered into a "black-hole" of neoliberal educational policies.

References

Almazabar, D. (2018). *Schools that rejected MINEDUC's school textbooks explain their reasons: "We take technical decisions not political ones."* Retrieved from: https://www.emol.com/noticias/Nacional/2018/02/20/895895/Inclusion-y-calidad-Directores-de-colegios-que-rechazaron-los-textos-escolares-del-Mineduc-explican-las-razones.html

Andrade-Molina, M. (2017). Incepted neoliberal dreams in school mathematics and the 'Chilean experience'. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth Congress of European Research in Mathematics Education* (CERME10) (pp. 1470–1477). Dublin, Ireland: DCU Institute of Education and ERME.

Benveniste, L. (2002). The political structuration of assessment: Negotiating state power and legitimacy. *Comparative Education Review, 46*(1), 89–118.

Cabalin, C. (2012). Neoliberal education and student movements in Chile: Inequalities and malaise. *Policy Futures in Education, 10*(2), 219–228.

Carrasco-Augilar, C., Ascorra, P., López, V., & Álvarez, P. (2018). Tensiones normativas de los fiscalizadores de la Superintendencia de Educación en la(s) políticas(s) de convivencia escolar en Chile. *Perfiles Educativos, XL*(159), 126–143.

Chubb, J., & Moe, T. (1990). *Politics, markets and America's schools*. Washington D.C., Brookings Institute.

Campos-Martínez, J., Corbalán, F., & Inzunza, J. (2015). Mapping neoliberal reform in Chile. In W. Au & J. Ferrare (Eds.), *Mapping Corporate Education Reform: Power and Policy Networks in the Neoliberal State* (pp. 106–125). New York-London: Routledge.

Escalante, F. (2015). *Historia mínima del neoliberalismo.* México: El Colegio de México.

Fan, L., Trouche, L., Qi, C., Rezat, S., & Visnovska, J. (Eds.). (2018). *Research on mathematics textbooks and teachers' resources: Advances and issues*. Cham: Springer.

Friedman, M. (1962). *Capitalism and freedom.* Chicago: The University of Chicago Press.

Gobierno de Chile. (2018). *Aprueba bases administrativas, bases técnicas, anexos y contrato tipo de licitación pública ID N°592-12-LS18, sobre servicio personal especializado de elaboración de contenido de textos escolares destinados a estudiantes y docentes de educación parvularia, básica y media de establecimientos subvencionados del país y derecho de uso no exclusivo, año 2020.* Santiago: Gobierno de Chile.

González, J., & Parra, D. (2016). Mercantilización de la Educación. Comentarios sobre la Reforma Educativa en Chile 2015. *Revista Enfoques Educacionales, 13*(1), 71–89.

Miñana, C. & Rodríguez, J. (2003). La educación en el contexto neoliberal. In D. Restrepo (Ed.), *La falacia neoliberal. Críticas y alternativas.* Bogotá: Universidad Nacional de Colombia.

MINEDUC (n.d.). *Textos Escolares.* Retrieved from: https://www.ayudamineduc.cl/ficha/textos-escolares-sige-4

MINEDUC (2017). *Ley de Inclusión Escolar.* Santiago: Gobierno de Chile.

MINEDUC (2016). *Decreto 53.* Retrieved from http://bcn.cl/1uwh0

Olivera, M. (2017). *Estudio de uso y valoración de textos escolares. Informe final.* Santiago: UNESCO.

Pais, A. (2012). A critical approach to equity. In O. Skovsmose & B. Greer (Eds.), *Opening the cage. Critique and politics of mathematics education* (pp. 49–91). Rotterdam: Sense.

Rojas, R. (2015). El negocio de los textos escolares en Chile. Retrieved from: https://radio.uchile.cl/2015/03/03/el-negocio-de-los-textos-escolares-en-chile/

EQUIVALENCE OF FRACTIONS IN 6TH GRADE BRAZILIAN TEXTBOOKS

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The concept of equivalence is elementary in the construction of the rational numbers. Hence, it should not be absent in the textbooks, and the students should be able to answer the question 'How can one decide if two given fractions a/b and c/d are equivalent?' and explain their answer. In this paper we report an analysis carried on in Brazilian textbooks focusing mainly on the questions: Is a complete characterization for equivalent fractions clearly presented to the students? Is equivalence used in comparison, addition and subtraction of fractions? Only partial characterizations were found, hence an important opportunity of developing students' mathematical thinking is lost. We also present a 'proof that explains' for the characterization for two given fractions to be equivalent which we consider is adequate for 6th grade students.

INTRODUCTION

The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School's classrooms and textbooks, and it should be well constructed with the students, in the sense that they are able to answer the question "How can one decide if two given fractions *a/b* and *c/d* are equivalent?" and explain their answer.

Considering that in Brazil the content fractions is resumed and carried on in the 6th grade, we decided to analyse 13 Brazilian textbooks from 4th to 7th grade focusing on the questions: i) how are fractions introduced and resumed? ii) How is equivalence treated? iii) Is any equivalence criterion present? iv) Is equivalence applied in comparison, addition and subtraction of fractions? We report in the present paper that analysis. The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like

Two given fractions a/b and c/d are equivalent if and only if ad=bc. (*)

In most cases only a partial equivalence criterion is presented, like

Two fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number. (**)

and it is based only on visualization, i.e., in a "see what happens" behaviour.

Believing that proofs should be present in the classrooms, as emphasised in the Topic Study Group 18 "Reasoning and Proof in Mathematics Education" of the International Congress on Mathematical Education (ICME 13), in 2016, we present some 'proofs

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 131–136). Paderborn: Universitätsbibliothek Paderborn.

that explains' (Hanna, 1990) which we consider are adequate for 6th grade students explaining the link among the different meanings of fractions (part-whole and quotient meanings) and also proving the complete equivalence criterion (*), since we could not find anything similar in any Brazilian textbook under analysis nor in any of the few textbooks from other countries that we had at hand.

THE RESULTS OF THE ANALYSIS

Table 1 shows the distribution of the 13 Brazilian textbooks under analysis.

Country	4th grade	5th grade	6th grade	7th grade
Brazil	2	2	7	2

Table 1. The distribution of the analysed Brazilian textbooks.

In all of them the concept of fraction is introduced/resumed by means of the so-called part-whole relation. Not much emphasis is given to equipartition, despite being very important for this subject. It would be advisable, in our opinion, to explore it, as in the we could find in an Italian textbook for the 3rd grade (see Figure 1).

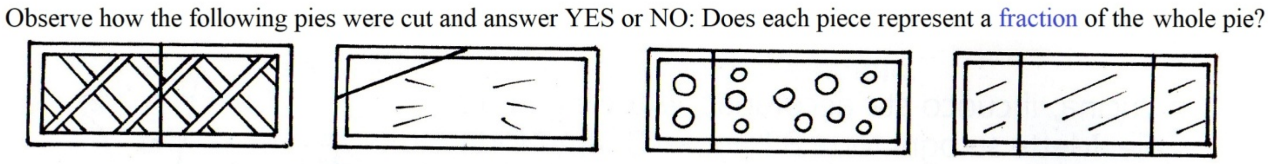

Figure 1. An exercise in Riccarddi (2008, p. 90) calling the attention for the idea of equipartition in order to be able to talk about fractions, translated by the authors.

No other meaning besides the part-whole relation was present in the analysed Brazilian textbooks. Nevertheless, it is clear that the link between the part-whole and the quotient meanings is a pre requisite for the decimal representation of a rational number represented by a fraction. It is interesting to mention at this point a French textbook for the 6th grade which makes it very clear that the quotient meaning of a fraction is an objective for this school year, but after resuming fractions and exploring the meaning of ¾ of the area of a surface, it only informs to the students "The quotient 35 : 11 is denoted by $\frac{35}{11}$ and is called a fraction." (Brault et al., 2005, pp. 94, 96). That is why we illustrate in Figure 2 an argument which explains this connection when we think of the division 2 : 3 as an equipartition, and which we find adequate for 6th graders.

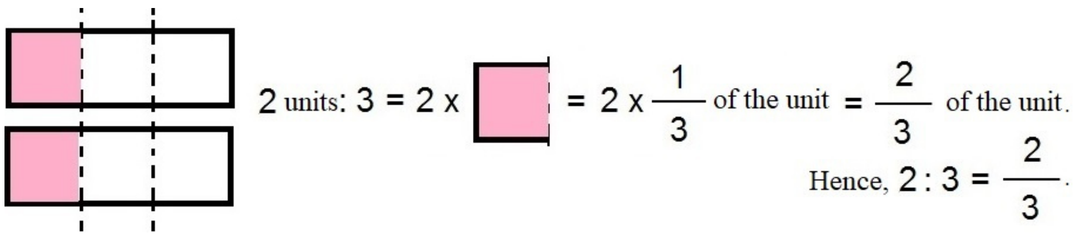

Figure 2. Explaining the link between part-whole and quotient meanings in case we think of the division as an equipartition.

In fact, it is shown there that, if we divide two units in three equal parts, then we get as a result two little parts which correspond, each of them, to one third of the given unit. Hence, the result of two units divided in three equal parts is two times one third of the unit, that is, two thirds (of the unit).

And in Figure 3 we illustrate an argument which explains the connection between the fraction 2/3 and the division 2 : 3 when we think of the division with the measure meaning. In this case, the divisor 3 should be considered the unit measure (in orange in Figure 3) with which one should measure the length of the yellow stripe. It is clear that the yellow stripe is 2/3 of the measure unit, hence the result of two units divided by three units is two, two thirds. of the measure unit

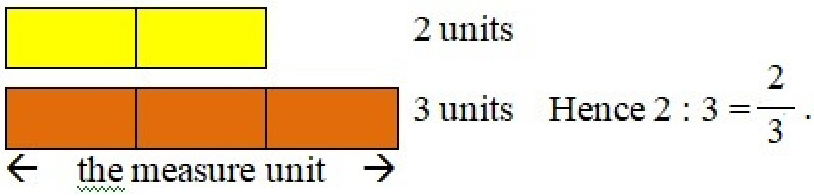

Figure 3. Explaining the link between part-whole and quotient meanings in case we think of the division as a measure.

The concept of equivalence appears already in the 5th grade Brazilian textbooks, however it is supported only by pictorial representations followed by a "see what happens" behaviour instead of being based on the definition of fraction. The same happens with four of the seven 6th grade Brazilian textbooks under analysis. No motivating exercise was found like the one found in the Italian textbook (Riccarddi, 2008) which exhibits different figures equipartitioned in 4, 6 and 8 parts and asks the 3rd grader to paint half of each one (Figure 4).

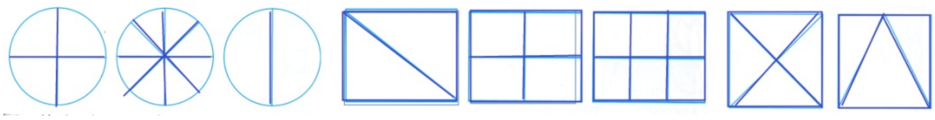

Figure 4. Part of an exercise in Riccarddi (2008, p. 91) asking the student to paint half of each picture.

It is our point of view that equivalence should be treated concomitantly with comparison, motivated by the following question, once two fractions are given: *Does the first fraction represent a quantity that is greater, smaller or equal to the quantity represented by the second one?* In fact, equivalence is only one of the possible cases in the so-called trichotomy law. Nevertheless, in six of the seven 6th grade Brazilian analised textbooks equivalence is discussed before comparing two non-equivalent fractions. In this case, it would be expected that equivalence would then be used to compare those fractions, but in four of these textbooks this strategy is not used and the arguments are supported only by pictorial representations suggesting the "see what happens" behaviour (Figure 5). It is interesting to mention that, till the present moment, we could only find one textbook, a Japanese one translated into Spanish

(Isoda & Cedillo, 2012, p. 24), where equivalence is motivated by the necessity of comparison of two fractions.

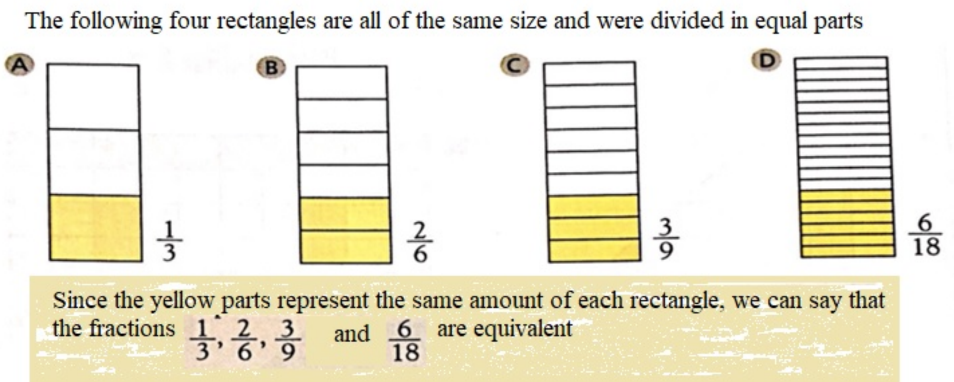

Figure 5. Definition of equivalent fractions without proving the assertion related to the same amount in Cavalcante, Sosso, Vieira, Poli (2006, p. 168, translated by the authors)

Another aspect related to equivalence is the following. Once fractions are introduced by means of the part-whole relation, the idea of quantity (of the unit) associated to the fraction is clear. Hence, since two fractions are called equivalent if they represent the same amount, it would be expected that the question "How can one decide if two given fractions *a/b* and *c/d* are equivalent?" would be considered when fractions are resumed in the 6th grade, with the purpose of consolidating the students' knowledge about fractions. Nevertheless, this question was not completely discussed in any of the 6th grade textbooks under analysis. But three of them called our attention. The first one, after giving a good example proving that 2/10 and 3/15 are equivalent, loses the opportunity of stating and proving the complete criterion "*Two fractions are equivalent if and only if one can transform one into the other by multiplying and/or dividing the numerator and the denominator by the same natural number*". In two other textbooks the complete criterion (*) is mentioned in exercises, but in the form of "prescribed rule" (Figure 6).

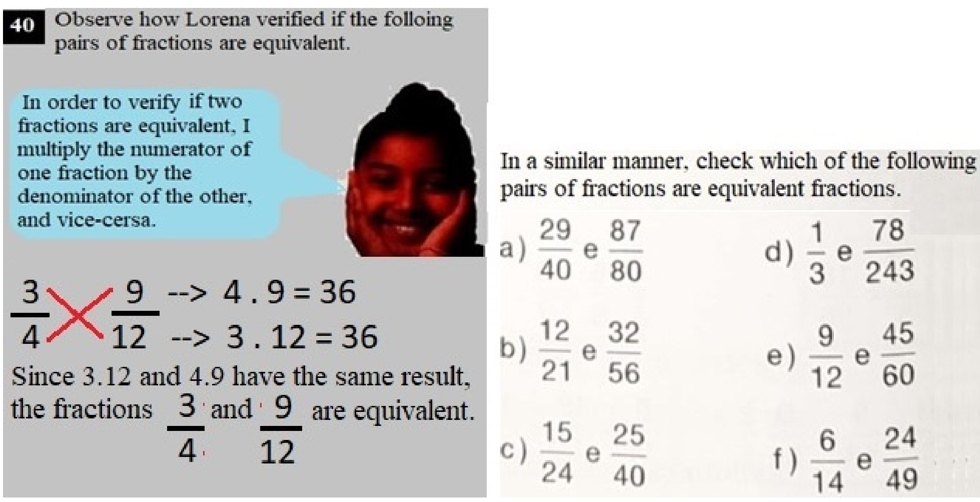

Figure 6. The equivalence criterion (*) in an exercise in Souza, Pataro (2012, p.136, translated by the authors)

All 6th grade analysed Brazilian textbooks solve addition and subtraction of fractions of different denominators using equivalent fractions. Nevertheless unnecessary emphasis is given to the minimum common multiple (see Figure 7) since any common multiple of the denominators of the given fractions can be used in order to compute the result of the given operation.

1º) $\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$ 6 is a multiple of 3 ⇒ lcm (6,3) = 6

2º) $\frac{2}{5} + \frac{3}{10} = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$ 10 is a multiple of 5 ⇒ lcm (5,10) = 10

Observe now some examples of fenominators in which one is not a multiple of the other.

1º) $\frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12} = \frac{2+3}{12} = \frac{5}{12}$ lcm (6,4) = 12

2º) $\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{6+5}{15} = \frac{11}{15}$ lcm (5,3) = 15

Figure 7. Excessive emphasis given to the least common multiple in addition (Bigode, 2015, p. 232, translated by the authors).

THE PROOF OF AN EQUIVALENCE CRITERION

In this section we present a "proof that explains" (Hanna, 1990) for the equivalence criterion (*) which we consider adequate for the 6th grade students. It is based on the fact that ther is no difficulty in comparing fractions with equal denominators. Hence, starting with the question *Do* 2/3 *and* 3/4 *represent the same quantity or not*? or *How could we compare the fractions* 2/3 *and* ¾? we can look for equivalent fractions with equal denominators arguing by means of a rectangular array (see Figure 8).

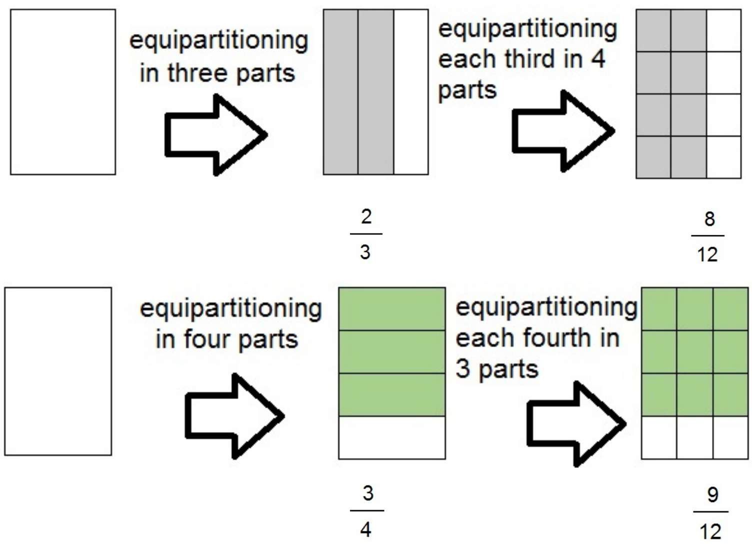

Figure 8. Looking for equivalent fractions with equal denominator.

Then the students can compare the original fractions after concluding that

$$\frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12} \quad \text{and} \quad \frac{3}{4}=\frac{3\times 3}{4\times 3}=\frac{9}{12}.$$

The next step is the generalization of this argument (in words, at least): we expect that, with the idea of a rectangular array in mind, the students are able to generalize the constructions in Figure 8 for any pair of fractions *a/b* and *c/d* and argue that they are equivalent if and only if *ad/bd* and *bc/bd* are equivalent, since

$$\frac{a}{b}=\frac{a\times d}{b\times d} \quad \text{and} \quad \frac{c}{d}=\frac{c\times b}{d\times b}.$$

But by the definition of fraction, *ad/bd* and *bc/bd* are equivalent if and only if their numerators coincide, since they have equal denominators. Hence *a/b and c/d are equivalent if and only if ad=bc.*

We call the readers attention to the fact that, not only the arguments in this proof helps to develop students' understanding about equivalence of fractions but also provide a technique which, initially supported by a visualization, can be used to compare, add and subtract any pair of fractions, developing as well the understanding of these topics.

FINAL COMMENTS

With the analysis of Brazilian textbooks carried on and reported in this work, it became clear that in countries like Brazil where emphasis is given on the study of fractions, it is essential to motivate and discuss a (complete) equivalence criterion. Among all the analysed textbooks, such a criterion was only explicitly mentioned in two Brazilian textbooks, however among the exercises and without any proof being required from the student. By stating and proving it in this text, we express our conviction that equivalence can be treated thoroughly as the theme fractions is resumed and deepened (6^{th} grade in Brazil) and is a good opportunity to develop students mathematical thinking.

References

Bigode, A. J. L. (2015). *Matemática do Cotidiano, 6º ano*. São Paulo: Scipione.

Brault, R., Daro, I., Perbos-Raimborg, D., Ploy, L., & Telmon, C. (2005). *Mathématiques – Collection Phare (6ᵉ)*. Paris: Hachette Éducation.

Cavalcante, L. G., Sosso, J., Vieira, F., & Poli, E. (2006). *Para saber Matemática, 5º série/6º ano*. São Paulo: Saraiva.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.

Isoda, M., & Cedillo, T. (2012) *Matemáticas para la Educación Normal* (Tomo IV, Vol. 1) México: Pearson Ed.

Riccarddi, C. (2008). *Guide Per la Scuola (3ª classe)*. Ancona: Ed. Raffaello.

Souza, J., & Pataro, P. M. (2012). *Vontade de saber Matemática, 6º ano*. São Paulo: FTD.

A COMPARATIVE ANALYSIS ON TRIGONOMETRY TEXTBOOKS FROM THREE COUNTRIES

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This paper presents an examination about mathematical textbooks for high school students from China, Japan and the United States at a macro level, with respect to general features, the trigonometric content included and the content sequence. Also, we try to build a macro level and micro level framework for textbook analysis on the basis of existing literature. Some findings are as follows: three countries value the trigonometry highly, but the Chinese textbook includes less trigonometric content than the other two countries. American and Chinese textbooks present trigonometric application after trigonometric function, while Japanese textbook presents these topics much earlier. Several suggestions are proposed at the end.

INTRODUCTION

D. F. Robitaille, the International Coordinator for TIMSS, once pointed out the importance of analysing textbooks, "In every country, mathematics textbooks exert a considerable influence on the teaching and learning of mathematics." (Howson, 1995, pp. 5–6). Called as "potentially implemented curriculum", textbooks help to build a bridge to connect the expected curriculum and implementation curriculum. ICME-10 (DG14) have focused on the development of textbooks in different countries, the role of textbooks in teaching and learning, and the research status of mathematical textbooks (Li, Zhang, & Ma, 2009). Nearly, Remillard et al. (2014) offered a conceptualization of the enacted curriculum and situates it within a curriculum policy, design, and enactment system, which emphasized the influence of instructional materials, including textbooks, on the curriculum reform. Therefore, there is no doubt about the importance of mathematical textbooks to students' learning and teachers' teaching. Li et al. (2009) pointed out that relevant efforts to examine mathematics textbooks have led to different research interests, including content topic inclusion, students' performance expectations, and content presentation and organization features. Although there are numerous studies on mathematics textbooks in different countries, most of them focus on the algebra content in elementary and secondary school textbooks (e.g., Fan et al., 2007; Li et al., 2009; Charalambous et al., 2010). While, there are several studies showing that students, pre-service mathematics teachers and even in-service mathematics teachers have some difficulties in understanding some knowledge points in trigonometry (e.g., Fi, 2006; Topcu et al., 2006). By focusing on the content topic of trigonometry, this study was designed to examine the general features, content topic inclusion and content sequence in selected textbooks from China, Japan, and the United States. In particular, we aimed to answer the following questions:

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 137–142). Paderborn: Universitätsbibliothek Paderborn.

1. What general features do selected textbooks have in trigonometry across the three education systems?

2. What content do selected textbooks include in the content topic of trigonometry for teaching and learning across the three education systems?

3. What presentation sequence do selected textbooks have across the three education systems?

THEORETICAL FRAMEWORK

Li et al. (2009) selected the fractional division to study the mathematics textbooks of China, Japan and the United States from the macro and micro levels. Similarly, Charalambous et al. (2010) established a two-dimensional framework including horizontal and vertical dimensions. Also, Óscar Chávez et al. (2011) described a process of development of assessment instruments that focused on evaluating American high school students' mathematics learning from two distinct approaches to content organization sequence, which showed the influence of content sequence on students' achievements. So, we compared the content of trigonometry in the three countries from the macro and micro levels. Macroscopically includes general features, the trigonometric content included and the content sequence. Microscopically, the framework of Yuan (2012) was referred to and simplified to analyse the role of unit circle when learning the trigonometry content.

METHODS

This textbook study is an outgrowth of a larger research project that aimed to investigate the identification, selection and arrangement of high school mathematics core content. Trigonometry textbooks from three countries were used in this study, including the regular high school curriculum standard experimental textbook series A (Referred as CTB) published by the people's education press of China, the new mathematics series (Referred as JTB) published by Japan and the algebra series (Referred as UTB) published by Prentice Hall of the United States. And then we will select the content of trigonometry in the textbooks and put them into three parts. We combine quantitative research with qualitative research. It presents the mathematical textbooks analysis framework of trigonometry, we carried out the textbook analysis at both the macro and micro levels (see Table 1).

Level	Aspect	Explanation
Macro level	General features	The appearance of teaching material, page number, colour, typesetting and other characteristics.
	Content topic inclusion	The chapter, unit and main content of trigonometry.
	Content sequence	The main content sequence of trigonometry

Micro level	Structure of trigonometric content	Use concept maps to analyse the breadth, depth, complexity, and interconnectedness of knowledge points
	Presentation features	Divided into 3 kinds of characteristic, including example and explanation, chart, exercise. Calculate the proportion that each characteristic occupies area.
	Expectations for students	Including problem background type, type of problem set and the cognitive level of the problems.

Table 1. Textbook analysis framework on Trigonometry content

RESULTS

Results show textbook variations in general features, content topic inclusion and content sequence across the selected three education systems.

General features

First of all, Chinese and American textbooks are in large format, while Japanese textbooks are designed in small format (13cm ×20cm).In terms of binding, Hardcover edition is adopted in American textbooks, while paperback edition is adopted in Chinese and Japanese textbooks. Secondly, the thickness of American textbooks far exceeds that of Chinese and Japanese textbooks. For example, the total number of pages of Chinese textbooks related to trigonometric content is 105.The American textbook contains 1049 pages, including 126 pages of trigonometry for 2 chapters. Finally, although Chinese and Japanese textbooks are partially colour, but just for the decoration title, column border, title page, photographs and so on. The background colours of Chinese textbooks cover and text notes are mostly light purple and light green. Japanese textbooks set colour photographs at the beginning of the title page and chapter. American textbooks are full of colourful pictures and pictures.

Content topic inclusion

Based on the classification of the content of "trigonometric functions" in the new curriculum standards in China, the content topic of trigonometry is divided into three modules: trigonometric functions and equations, trigonometric identities, and the application of trigonometry. At the same time, we determine 19 main contents to cover all trigonometry knowledge (Table 2 and Table 3). Trigonometry textbooks of the three countries basically covers the content of any angle, radian system, trigonometric functions, the law of sines and cosines, etc., which shows that comparative analysis is feasible. As seen from Table 2, in terms of content quantity, the numbers of content involved in Japanese and American textbooks are the same (both are 15 items), while the number of main contents in Chinese textbooks is the least (11 items).

Module	Numbers	CTB	JTB	UTB
Trigonometric functions and equations	13	6	10	11
Trigonometric identities	3	2	3	2
The application of trigonometry	3	3	2	2
Total	19	11	15	15

Table 2. The trigonometric content included across three education systems

According to Table 3, Chinese textbooks mainly lack acute triangle ratio, inverse trigonometric function, "secant, cosecant, cotangent", "triangular equation" and other contents. The main reasons are as follows: (1) the Chinese students who use the textbook of "A version" have learned the "acute triangle function" in junior high school, which is equivalent to the "triangle ratio" in Japanese textbooks. And (2) after the curriculum reform, the new curriculum standard tries to simplify the content and reduce students' academic burden, so the "cotangent", "find the angle when known trigonometric function values" are deleted. Although Japanese and American textbooks have the same the number of content, the specific content is slightly different. The United States has "inverse trigonometric function", "secant, cosecant, cotangent function", and Japanese textbooks do not cover the above content. JTB attaches great importance to the connotation of "triangle ratio" and promotes it, and has its own section of "triangle inequality". In the United States, the "acute triangle ratio" is just regarded as the definition of sines and cosines in a right triangle and a tool to solve the triangle, rather than as a "function".

Module	Main content knowledge	CTB	JTB	UTB
Trigonometric functions and equations	Trigonometric Ratios for Acute Angle		✓	✓
	Expansion of Trigonometric Ratios		✓	
	Any angle and Radian measure	✓	✓	✓
	Conception of trigonometric functions	✓	✓	✓
	Relationship between trigonometric functions of the same angle	✓	✓	✓
	Induction formulas of trigonometric functions	✓	✓	✓
	Graphs and properties of trigonometric functions	✓	✓	✓
	Transformation for the Graphs of trigonometric functions	✓		✓

Graphs and properties of inverse trigonometric functions		✓
Graphs and properties of secant, cosecant and cotangent functions		✓
Periodic function		✓
Trigonometric equations	✓	✓
Trigonometric inequalities	✓	

Table 3. Main content of trigonometric functions and equations

Content sequence

Next, we will take the concept of trigonometric functions as an example to further analyse the content arrangement sequence of mathematics textbooks across the three countries, focus on the introduction process of the concept of trigonometric functions, and observe the role of unit circle in learning and teaching trigonometric functions. For example, trigonometric knowledge in Chinese and Japanese textbooks is arranged according to the ideas from acute trigonometric ratios to trigonometric functions for any angle, the sequence follows the cognitive order from special to general. American textbooks, on the other hand, arrange them according to the properties of periodic functions, trigonometric functions and acute trigonometric functions in a cognitive order from general to special. In Chinese and Japanese textbooks, the item "periodic function" is just mentioned as a superordinate concept of trigonometric functions, while American textbooks discuss them in detail. Starting from the common periodic phenomena such as electrocardiogram, alternating current and sound wave, after students have an overall perception of periodic movement, the trigonometric function is derived, and then acute triangle ratio is learned in right triangle, which belongs to inferior learning.

In addition, American and Chinese textbooks present trigonometric application after trigonometric function, while Japanese textbook presents these topics much earlier. That is to say, Since the Angle range of the sine and cosine laws is between 0° and 180°, and trigonometric functions has no influence on them, the sine and cosine laws may be learnt earlier than the trigonometric functions. Therefore, different arrangement ideas may lead to differences in content selection.

DISCUSSION

On the whole, selected textbooks from three countries are examined at a macro level. Some results are as follows: these three countries value the trigonometry highly, but the Chinese textbook includes less trigonometric content than the other two countries because of education policy. American and Chinese textbooks present trigonometric application after trigonometric function, while Japanese textbook presents these topics much earlier. In addition, we believe that the unit circle should be used as a powerful tool in trig function learning. For example, students can use the symmetry of the unit circle to explore the induction formula of trigonometric functions. There are still many issues to be discussed in the future, such as similarities and differences at the micro level. The cross-system differences shown in textbooks suggest the nature of mathematics content ,which may be treated differently in textbooks cross-nationally (Li et al., 2009). The similarities and differences in different countries on textbooks reflect the diversity in social culture and functional positioning of textbooks. A reasonable and effective interpretation of these differences can help us better develop textbooks that are conducive to mathematical learning.

References

Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning, 12*, 117–151.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: a comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics, 66*, 61–75.

Fi, C. (2006). Pre-service secondary school mathematics teachers' knowledge of trigonometry: co-function. In S. Alatorre, J. L Cortina, M. Sáiz, & A. Méndez (Eds.). *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Mérida, México: Universidad Pedagógica Nacional.

Howson, G. (1995). *Mathematics textbooks: A comparative study of grade-8 texts.* Vancouver, Canada: Pacific Education Press.

Li, Y., Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: the case of fraction division. *ZDM Mathematics Education, 41*(6), 809–826.

Li, Y., Zhang, J., & Ma, T. (2009). Approaches and practices in developing school mathematics textbooks in China. *ZDM Mathematics Education, 41*, 733–748.

Chávez, O., Papick, I., Ross, D., & Grouws, D. A. (2011). Developing fair tests for mathematics curriculum comparison studies: The role of content analyses. *Mathematics Education Research, 23*, 397–416.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education*, *46*(5), 705–718.

Topçu, T., Kertil, M., & Akkoç, H. (2006). Pre-service and in- service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková, (Eds.), *Proceedings of the 30th Conference of the international group of the Psychology of Mathematics Education* (Vol. 5, pp. 281–288).

Yuan, S. (2012). *A comparative study on trigonometry in textbooks from China, Japan, the United States and Britain.* Shanghai: East China Normal University.

AN ANALYSIS OF TEACHING METHODS OF DIVISION OF FRACTIONS IN SOUTH KOREA ELEMENTARY MATHEMATICS TEXTBOOKS

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The content of Korean elementary school math textbooks before and after the 2015 revision were analysed and compared to present the trend of teaching method of 'division of fractions' in South Korea. Fraction division is typically solved by using 'invert and multiplying' algorithm. It is easy for students to calculate mechanically without understanding. But, in fact, it is challenging for students to understand why they should use 'the invert' in fraction division. Fraction division require understanding of computational processes using complex fraction concepts, so students' understanding will vary depending on what order they present them and what kind of models they use. South Korea is one of the countries that maintain the higher achievement in international assessments such as PISA and TIMSS, using a single state-authored textbook that applies to the national curriculum. Also, Korea recently revised its curriculum in 2015. In this study, we compared two series of Korean math textbooks that were written with the 2009 and 2015 curriculum. To provide a framework of the contents of fraction division, we focus on the 'fraction division conceptualization,' 'visual model' and 'algorithm' used in textbooks that were analysed with the changes in Korea elementary school math textbooks.

INTRODUCTION

The fraction division (FD) is one of the difficult concepts to understand for both students and teachers. Several studies focus on teachers' pedagogical content knowledge and students' misconception about fraction division (Ma, 1999; Tirosh, 2000; Adu-Gyampi et al., 2019). Prospective teachers and students often misunderstand the algorithm of fraction division or present inappropriate representations. Considering that textbooks are the major resources for teachers and students in the classroom (Reyes, Reys, & Chavez, 2004), it is important to study how the textbooks present the division of fractions with real-world problems that reflect the real context and conceptual models.

South Korea is one of the countries that maintain the high achievement on mathematics in international assessments such as Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) (Mullis, Martin, & Hooper, 2016; OECD, 2016). South Korea revised mathematics curriculum in 2015 and emphasizes the development of six mathematical competencies. There was an emphasis of three mathematical process in the 2009 revised curriculum: problem solving, reasoning, and communication. The 2015 revised curriculum added 'creativity and convergence, information processing, and attitudes and practice' to help students become interested and confident in math (KOFAC, 2019b). Korea use a state-authored elementary mathematics textbook series that is based on the national curriculum. In

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this study, we analysed the contents and presentation of the 'divisions of fractions' unit in two series of elementary school math textbooks: one with the 2009 curriculum and the other based on the 2015 curriculum to compare changes in contents of textbooks.

THEORETICAL FRAMEWORK

Conceptualization of fraction division

Adu-Gyamfi et al. (2019) mentioned that the division of fractions is an essential mathematical domain, and emphasized the need for teachers to interpret the concept of FD in various ways and to use them in contexts. Table 1 presents the five conceptualizations of FD adopted from Sinicrope et al. (2002), the context, linguistic representation, and algorithms in which were used (Adu-Gyamfi et al., 2019, p. 4).

Conceptualization	Contextual example Verbal representation	Algorithm Algebraic representation
Measurement division (Part-whole or measure)	(c/d) of an ounce of gold is needed to create an earring. How many earrings can be created with (a/b) ounces of gold?	$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{be}{bd} = ad \div bc$
Partition division (Part-whole or measure)	(a/b) pounds of candy is shared equally among (c) friends. How much candy does each friend receive? (c divides a)	$\frac{a}{b} \div c = \frac{a \div c}{b}$
	(c) friends bought (a/b) kilogram of chocolate and shared it equally. How much chocolate did each person get? (c does not divide a)	$\frac{a}{b} \div c = \frac{a}{b \times c}$
Determination of a unit rate (ratio)	Rose used (a/b) of a can of frosting to frost (c/d) a cake. How much of the cake could she frost with the whole can of frosting?	$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$
Division as the inverse of an operator multiplication (operator)	In a seventh-grade survey of lunch preferences, (a) students said they prefer pizza. This is (c/d) times the number of students who prefer the salad bar! How many prefer the salad bar?	$a \div \frac{c}{d} = \frac{d}{c} \times a$
Division as the inverse of a Cartesian product (ratio)	The probability of rolling a five with a fair, hexahedral die is (c/d) and the probability of rolling a five and spinning an even number is (a/b). What is the probability of spinning an even number?	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\left(\frac{cd}{cd}\right) \div \frac{c}{d} = \frac{acd \div c}{bcd \div d} = \frac{ad}{bc}$

Table 1. Fraction division conceptualizations (adapted from Adu-Gyamfi et al., 2019, p. 4)

Visual models for fraction

Appropriate visual models for fractions are essential in the learning and understanding of fractions and operations. Three types of fraction models are widely used: area (region), length, and set models (Van de Walle et al., 2008, p. 288). These fraction models are used in various concepts of fractions, such as equal fractions, as well as in the size comparison and the operations of fractions (Van de Walle et al., 2008, pp. 293–325).

<u>The area model</u> is used when describing the part-whole concept of fractions using circles, rectangles, grid patterns, pattern blocks, etc. Because it uses concrete materials or a picture with an area, it is often used to reveal the meaning of equal division or sharing task. <u>The length model</u> is a more frequently used one in measurement situations using Cuisenaire rod, number lines, and paper straps, etc. <u>The set model</u> represents the whole and part with

several discrete quantities as a set (Van de Walle et al., 2008, pp. 288–291). However, the meaning of the operation should be taken into consideration when addressing the arithmetic operation of the fractions. When using these models, it is important to understand the context of the problem and make appropriate representations.

The research questions are as follows:

1. How do the fraction division contents (learning topics and algorithms) of Korean 2009 textbooks and 2015 textbook change?

2. How do the word problems and the visual model used in the Korean 2009 and 2015 textbooks interpretation the fraction division and how do they change?

METHOD

Selection of textbooks and activities

In Korean textbooks, the topic of FD was divided into two parts: 'Dividing fractions by whole number' and 'Dividing fractions by fraction'. The 2015 textbook includes the first part in the first semester of the 6th grade textbook, the second part will appear in the second semester textbook although it has not been published at the time of the study (KOFAC, 2019a). To fairly compare how students learn the same concepts of FD between the two series of the textbooks, we exclude the second part of FD unit from the 2009 textbook.

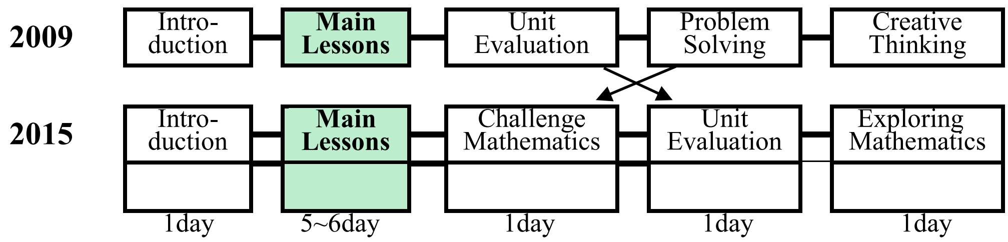

Figure 1. A comparison of unit components of 2009 and 2015 textbooks

To select sections to be analysed, we searched for the composition of the unit in Korean elementary school textbooks. The teacher guidebook specifies how teachers should plan to teach each unit, consisting of five main parts (Figure 1). The introduction section presents illustrations or illustrations related to the content, which is linked to the content of the main lessons. Main lessons usually include lesson topics for 5-6 days and other sections in the unit are intended for one day each. Each of main lessons of both textbooks offers the learning topic, one opening activity, 3-4 activities, and one finish activity in 2009 textbooks. The slight modification on the 2015 textbooks is to present the activities only with numbers without naming opening and finish activities. (KOFAC, 2015b, p. 35; KOFAC, 2019b, p. 45).

We are interested in understanding how fraction division concepts are introduced, this study is focused on the activities in the main lessons, not evaluation and problem solving. For this reason, we only analysed the activities in the main lessons. We also exclude the last activity of each of main lessons as it is intended for students to practice or apply the concepts, not learning for the first time. There are total of 22 activities in the second semester of the fifth grade of the 2009 textbook, and 15 activities from the 2015 textbook.

Analytic framework

In this paper, we compared contents of FD in textbooks with the conceptualization of FD in Table 1 used by Adu-Gyampi et al. (2019) as the analytical framework. To answer the first question, we compared the learning content through the algorithms covered by the study topic (see columns under *topic* in Figure 2) and activities presented in each main lesson. For the second question, we analysed word problems and the visual models of each activity to investigate how the textbooks interpretation the FD. The word problems were analysed using the FD conceptualization framework (see Table 1), and the visual models were categorized into areas, lengths, and set models to find out how they are used for interpretation FD concepts (Adu-Gyamfi et al., 2019, p. 4; Van de Walle et al., 2008, p. 288).

RESULTS

The content of textbook - topics and algorithms

Types of FD	2009 textbook		2015 textbook	
	topic	**Algorithm Algebraic representation**	**topic**	**Algorithm Algebraic representation**
(Whole number) ÷ (Whole number)	Representing ‘(Whole number)÷(Whole number)’ as a multiplication	$a \div b = a \times \frac{1}{b}$	Representing quotient of ‘(Whole number)÷(Whole number)’ as a fraction (1)	$a \div b = \frac{a}{b}$
	Representing quotient as fraction	$a \div b = \frac{a}{b}$	Representing quotient of ‘(Whole number)÷(Whole number)’ as a fraction (2)	$a \div b = \frac{a}{b}$ $(a>b)$
(Fraction) ÷ (Whole number)	Calculating ‘(proper fraction)÷(Whole number)’	$\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$ $(a<b)$	Let’s find out ‘(fraction)÷(Whole number)’	$\frac{a}{b} \div c = \frac{a \div c}{b}$ $(a=ck)$
	Calculating ‘(improper fraction)÷(Whole number)’	$\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$ $(a>b)$	Representing ‘(fraction)÷(Whole number)’ as a multiplication	$\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$
	Calculating ‘(mixed number)÷(Whole number)’	$a\frac{c}{b} \div d = \frac{(ab+c)}{b} \times \frac{1}{d}$	Let’s find out ‘(mixed number)÷(Whole number)’	① $a\frac{c}{b} \div d = \frac{(ab+c) \div d}{b}$ ② $a\frac{c}{b} \div d = \frac{(ab+c)}{b} \times \frac{1}{d}$

Figure 2. Contents structure of Unit of Fraction Division in 2009 and 2015 textbook

Figure 2 is a comparison of the main lesson content each unit of FD between the 2009 and the 2015 textbooks. It is organized by how FD topics are introduced, and types of algorithms suggested to use in each main lesson. Figure 2 presents that both textbooks represent two types of FD topics: ‘(whole number)÷(whole number)’ type and ‘(fraction)÷(whole number)’type, and three types of algorithm: (a) representing quotient as fraction, (b) invert and multiplying, and (c) the numerator of the dividend is multiple of the divisor.

The results confirm that 2009 textbooks emphasize the procedures of algorithms, and 2015 textbooks focus on the various expressions. It is supported by the type of algorithm in each of the two FD types. Firstly, as the ‘(whole number)÷(whole number)’ types of FD, 2009 textbook presented the invert and multiplying algorithm first, with emphasis on procedural processes. On the other hand, the 2015 textbook only suggested representing quotient as fraction algorithm. This shows that main concern has shifted from focusing on FD algorithm procedures to expressing the results of the quotient of natural numbers into different fractions (proper fractions, improper fractions and mixed number). Secondly, as the '(fraction)÷(whole number)' types of FD, 2009 textbook only presented invert and multiply algorithm, while 2015 textbook suggested two types of algorithm: the invert and multiplying, and the numerator of the dividend is multiple of the divisor. This suggests that the FD can be thought in different ways depending on the situation.

The word problems and the visual models

There is another evidence that 2009 textbook emphasis on procedural processes of algorithm while 2015 textbook focus on the various representations and intuitive understanding of the students.

For the word problem, in the 2009 textbook, only two activities are presented in 'partition division' concept, which only suggested in the '(whole number)÷(whole number)' type of FD. For the visual models, the 2009 textbook presents 14 activities suggested to use visual models: 1 activity with a set model, 8 activities with an area model, and 3 activities with a length model. First, in '(whole number) ÷ (whole number)', 3 activities with the length models and 1 activity with the set model are used. The length models present the invert-and-multiply algorithm, and the set model is presented to relate one to the fraction expression, '1 is 1/4 of 4'. Second, most of the activities considering '(fraction) ÷ (whole number)', consist of rectangular area models and a formula, which leads to the invert-and-multiply algorithm. On the other hand, 2015 textbook provides different mathematics concepts. There were six word problems, 4 activities used 'partition division', 1 activity used 'division as the inverse of an operator multiplication,' and 1 activity with the 'division as the inverse of a Cartesian product.' The latter two word problems were presented in the last main lesson titled, 'Let's find out '(mixed number) ÷ (Whole number)'. In addition, unlike 2009 textbook, 2015 textbook presents area models for '(whole number) ÷ (whole number)' first, length models for '(fraction) ÷ (whole number)' which includes 'double number line', and the rectangular area model and number line to help students understand each algorithm intuitively.

CONCLUSION AND DISCUSSION

This study described changes in FD concepts of Korean 2009 and 2015 math textbooks. The results are presented affective characteristics emphasized in the key competency of the 2015 revised curriculum is also emphasized in the FD contents of 2015 textbook. Both 2009 and 2015 textbooks presented the FD into 'partition division' with different types of algorithms, word problems and visual models. The 2009 textbook intended to present the 'invert and multiply' algorithm logically and procedurally. On the other hand, the 2015 textbook was intended to arouse students' interest with various representation, such as algorithms, and intuitive visual models.

Since we only examined the first part of the FD units of textbooks (recall that the second part of 2015 textbooks have not been published yet), the results of this study show that all activities presented in FD main lessons only require students to use whole numbers for divisors. The study indicates a number of further areas for research. First, the need of the 'invert and multiplying' process presented in '(whole number) ÷ (whole number)' of the 2009 textbook, which was deleted from the 2015 textbook. Second, the relevance to the linguistic representation '1 is 1/4 of 4' presented in the set model and similar representation of 'division as the inverse of an operator multiplication' (a is how many times of b). Third, contents of '(fraction) ÷ (fraction)' represented in the second semester of 2015 textbooks that have not yet been published. And the logical linking to this study, '(fraction) ÷ (whole number)' FD contents.

References

Adu-Gyamfi, K., Schwartz, C. S., Sinicrope, R., & Michael J. B. (2019). Making sense of fraction division: Domain and representation knowledge of preservice elementary teachers on a fraction division task. *Mathematics Education Research Journal*, online first, 1–22.

Korea Foundation for the Advancement of Science and Creativity [KOFAC]. (2015a). *Mathematics 5–2*. Seoul: Chunjae Education.

Korea Foundation for the Advancement of Science and Creativity [KOFAC]. (2015b). *The guidebooks for teacher mathematics 5–2*. Seoul: Chunjae Education.

Korea Foundation for the Advancement of Science and Creativity [KOFAC]. (2019a). *Mathematics 6–1*. Seoul: Chunjae Education.

Korea Foundation for the Advancement of Science and Creativity [KOFAC]. (2019b). *The guidebooks for teacher mathematics 6–1*. Seoul: Chunjae Education.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics.* Chestnut Hill: TIMSS & PIRLS International Study Center, Boston College.

OECD. (2016). *PISA 2015 results (Volume I): Excellence and equity in education*. Paris: OECD Publishing.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership: Journal of the Department of Supervision and Curriculum Development N.E.A, 61*(5), 61–66.

Sinicrope, R., Mick, H. W., & Kolb, J. R. (2002). Interpretations of fraction division. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 153–161). Reston: National Council of Teachers of Mathematics.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5–25.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2008). *Elementary and middle school mathematics: Teaching developmentally.* Boston, MA: Allyn and Bacon.

THE LEARNING OF DOUBLE INTEGRAL CONCEPT USING A TEXTBOOK

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In this paper we direct our attention to students' inscriptions when using a differential and integral calculus textbook to learn the double integral concept. It will be shown that certain kind of inscriptions are diagrams, in Peirce's semiotic sense, which are valuable means to learn the double integral concept. We analyzed the students' diagrams produced during an integral and differential calculus lecture of an engineering course at a Brazilian university. The students began the study of the double integral using the textbook Calculus vol 2 by James Stewart. Our findings denote that the way the textbook introduces the double integral concept favors the students' diagrammatic reasoning. The students' invented and transformed diagrams designed while using the textbook had influenced their learning process.

INTRODUCTION

Textbooks offer opportunities to learn for both- students and teachers- and from this point of view, the textbook is a support for calculus lectures, both for a preliminary students' reading to indicate how a particular study will be developed, and as the main resource for professor to teach (Almeida & Silva, 2018).

It is not surprising that textbooks play a crucial role for the construction of mathematical knowledge through ordering, presenting and explaining of mathematical concepts and problems (O'Halloran, 2018). The mathematical signs used in the textbook are mainly seen as instruments for coding and describing the mathematical objects (i.e. concepts) for operating with these objects, communicating mathematical knowledge to professors and students. In fact, Mathematics is a science which concerned in inventing and using signs and the mathematical knowledge is constructed using language, images and symbols. In this way Peirce's semiotic can be introduced as an instrument for describing aspects of learning mathematics by using a calculus textbook.

In this paper we focus on the students' inscriptions when they use a differential and integral calculus textbook to learn the concept of double integral. It will be shown that certain kinds of inscriptions are diagrams, in Peirce's semiotic sense. We analyzed students' diagrams of an integral and differential calculus lecture in an engineering course at a Brazilian university. In order to introduce the double integral concept a preliminary study was done by students using the textbook *Calculus vol 2* by James Stewart. Following this study, the teacher asked them to solve a problem which resolution proffers a possibility to introduce the double integral concept.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 149–154). Paderborn: Universitätsbibliothek Paderborn.

A SEMIOTIC PERSPECTIVE FOR THE STUDENTS' LEARNING: THE DIAGRAMMATIC REASONING

The different semiotics theories agree that signs are basically means of signifying an object or means of representing something for somebody. With regard to the epistemologically based semiotics by Charles S. Peirce, a central point is the emphasis on other fundamental functions of signs: signs as means of thought, of understanding, of reasoning and of learning (Hoffman, 2005). Peirce introduced a far-reaching project to demonstrate the importance of signs by emphasizing this point.

In this paper we shall refer only on some details of this semiotic approach. Particularly, we will refer to Peirce's concept of *diagrammatic reasoning*. Hoffmann (2005) argues that with this concept one can explain the development of knowledge basing on a three-step activity: constructing signs, experimenting with them, and observing the results.

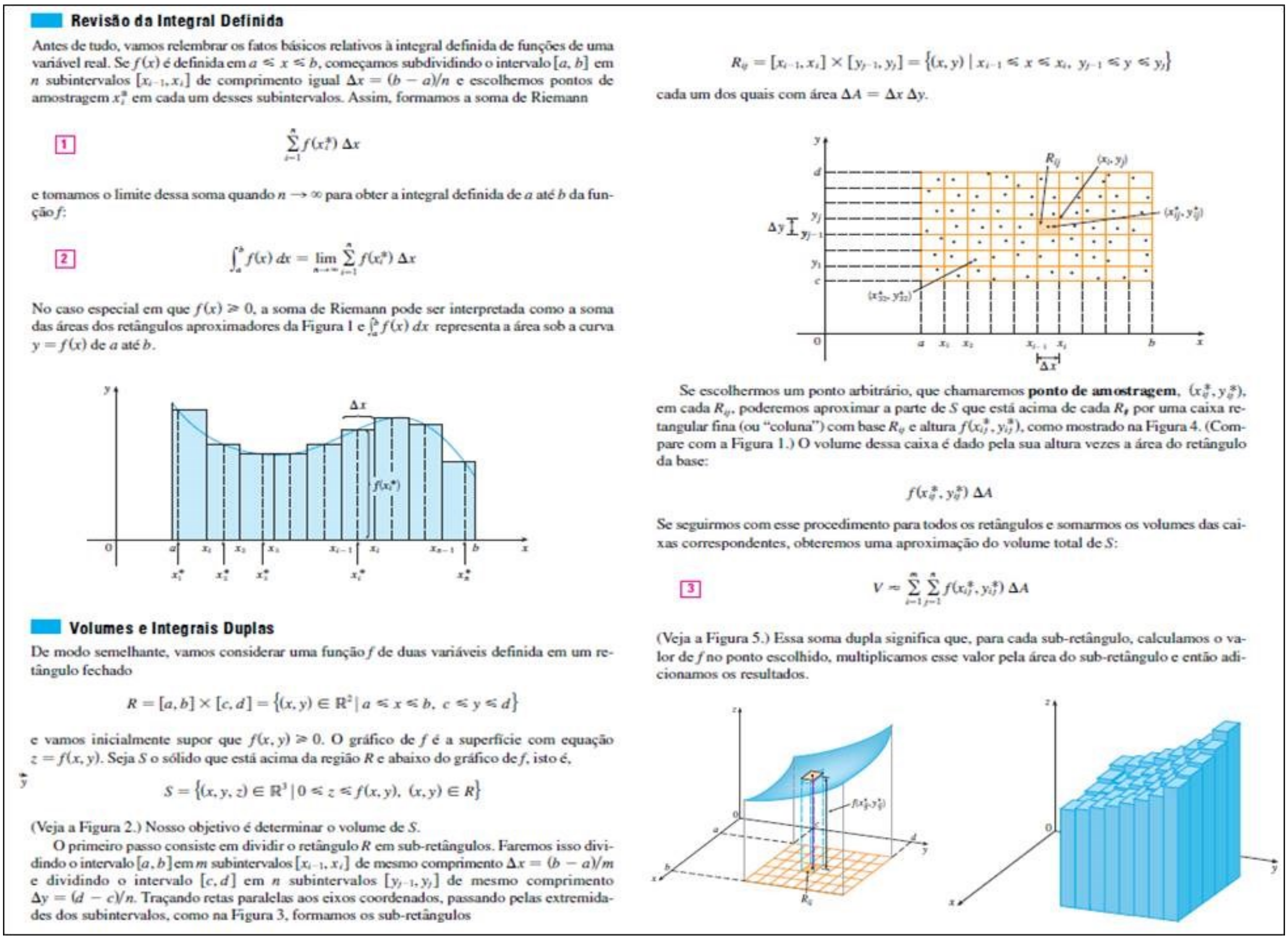
In order to understand what Peirce means by *diagrammatic reasoning*, we need to know something about what he called *diagram.* Peirce defines a diagram as a "representamen which is predominantly an icon of relations and is aided to be so by conventions" (Hoffmann, 2005 p. 46). Although Peirce also refers to diagrams as indices or symbols it is the iconic character of diagrams which is the most important. Therefore, we can say that diagrams are icons which are constructed following certain rules and may thereby show relations and must be carried out upon a consistent system of representation. The use of diagrams as a special kind of icon makes it possible to perform experiments when learning mathematics. By representing a problem using a diagram, people can experiment their cognitive meanings, make experiments and construct new knowledge which is called *diagrammatic reasoning.*

Kadunz (2016) identified that when students learning mathematics by means of this reasoning, firstly they have to construct a diagram (for example: an equation of algebra, a geometrical drawing by pencil or software, a graph to solve a problem). When this construction is finished, they can start experimenting. For example, a graph may be associated with an algebraic equation, or they can construct a graph on a three-dimension space using what they already know about graphic representations on the two-dimensional plane. Finally, on a third step, the results of the experiment may be explored and new relations can be noticed or visualized (Kadunz, 2016; Hoffmann, 2005). It is just in this way that Kadunz (2016) refers to Peirce's assertion that a diagram constructed by a mathematician "puts before him an icon by the observation of which he detects relations between the parts of a diagram other than those which were used in its construction" (Kadunz, 2016 p. 119). Taking into account this remarks about diagrams and diagrammatic reasoning we turn our attention to students' learning regarding the double integral concept when they use a calculus textbook.

THE STUDENTS ACTIVITY USING THE TEXTBOOK

The research we refer to in this paper was carried out in 2018 to a class of 35 students in a differential and integral calculus lecture of an engineering course at a Brazilian university. We refer to an activity which presents the introduction of double integral concept using the textbook *Calculus* vol 2, 7th edition written by James Stewart (Portuguese translated version). Firstly, the teacher asked students to study the introduction of double integral concept at the textbook (Figure 1) and after that the activities were guided by the teacher in the class.

The introduction of the double integral concept in the textbook begins with a revision of the integral geometric concept of a one variable function. This approach is broadened with the purpose of presenting how we can study the volume of a solid delimited by a two variables function defined in a closed region R.



Revisão da Integral Definida

Antes de tudo, vamos relembrar os fatos básicos relativos à integral definida de funções de uma variável real. Se $f(x)$ é definida em $a \leq x \leq b$, começamos subdividindo o intervalo $[a, b]$ em n subintervalos $[x_{i-1}, x_i]$ de comprimento igual $\Delta x = (b-a)/n$ e escolhemos pontos de amostragem x_i^* em cada um desses subintervalos. Assim, formamos a soma de Riemann

1 $$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

e tomamos o limite dessa soma quando $n \to \infty$ para obter a integral definida de a até b da função f:

2 $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

No caso especial em que $f(x) \geq 0$, a soma de Riemann pode ser interpretada como a soma das áreas dos retângulos aproximadores da Figura 1 e $\int_a^b f(x)\,dx$ representa a área sob a curva $y = f(x)$ de a até b.

Volumes e Integrais Duplas

De modo semelhante, vamos considerar uma função f de duas variáveis definida em um retângulo fechado

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

e vamos inicialmente supor que $f(x, y) \geq 0$. O gráfico de f é a superfície com equação $z = f(x, y)$. Seja S o sólido que está acima da região R e abaixo do gráfico de f, isto é,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$

(Veja a Figura 2.) Nosso objetivo é determinar o volume de S.

O primeiro passo consiste em dividir o retângulo R em sub-retângulos. Faremos isso dividindo o intervalo $[a, b]$ em m subintervalos $[x_{i-1}, x_i]$ de mesmo comprimento $\Delta x = (b-a)/m$ e dividindo o intervalo $[c, d]$ em n subintervalos $[y_{j-1}, y_j]$ de mesmo comprimento $\Delta y = (d-c)/n$. Traçando retas paralelas aos eixos coordenados, passando pelas extremidades dos subintervalos, como na Figura 3, formamos os sub-retângulos

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$

cada um dos quais com área $\Delta A = \Delta x\,\Delta y$.

Se escolhermos um ponto arbitrário, que chamaremos **ponto de amostragem**, (x_{ij}^*, y_{ij}^*), em cada R_{ij}, poderemos aproximar a parte de S que está acima de cada R_{ij} por uma caixa retangular fina (ou "coluna") com base R_{ij} e altura $f(x_{ij}^*, y_{ij}^*)$, como mostrado na Figura 4. (Compare com a Figura 1.) O volume dessa caixa é dado pela sua altura vezes a área do retângulo da base:

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Se seguirmos com esse procedimento para todos os retângulos e somarmos os volumes das caixas correspondentes, obteremos uma aproximação do volume total de S:

3 $$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

(Veja a Figura 5.) Essa soma dupla significa que, para cada sub-retângulo, calculamos o valor de f no ponto escolhido, multiplicamos esse valor pela área do sub-retângulo e então adicionamos os resultados.

Figure 1. The double integral concept introduction (Stewart, p. 874–875)

After the preliminary study using the textbook students were asked by the teacher to solve a problem proposed in this book (Figure 2). However, it was not allowed students to use the textbook while they were solving the problem.

For about 60 minutes students dedicated themselves to construct a representation of the R region at the two-dimensional plane. Then they evaluated the volume of each of the approximating rectangular boxes as it is suggested in Figure 1. In this paper we refer to the inscriptions and dialogues of three students and we will call them A, B and C. We selected these three students because they participated in all activities, including the interview after the activities development.

EXEMPLO 1 Estime o volume do sólido que está acima do quadrado $R = [0, 2] \times [0, 2]$ e abaixo do paraboloide elíptico $z = 16 - x^2 - 2y^2$. Divida R em quatro quadrados iguais e escolha o ponto de amostragem como o canto superior direito de cada quadrado R_{ij}. Faça um esboço do sólido e das caixas retangulares aproximadoras.	Example 1- Estimate the volume of the solid that is above the square R = [0,2] × [0,2] and below the elliptical paraboloid $z = 16-x^2-2y^2$. Divide R into four equal squares and choose the sampling point as the upper right corner of each square R. Draw a graph of the solid and of the approximate rectangular boxes.

Figure 2. The problem to be solved by students (Stewart, 2013, p. 887)

In order to solve the problem, initially the students made efforts to represent the statement: "Divide R into four equal squares and choose the sampling point as the upper right corner of each square R_{ij}" (Figure 3). The inscriptions they constructed and that we present in Figure 3 enabled the students to obtain the volume of the solid by using these four approximating boxes as requested on the problem statement.

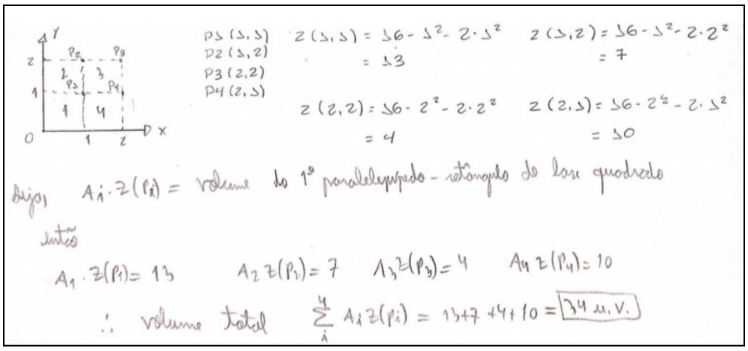


Figure 3. Student's inscriptions

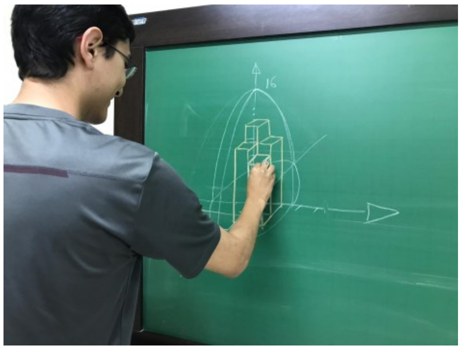


Figure 4. Student's graphical representation

When this first solution had finished, one of the students asked the teacher to sketch a graphical representation on the blackboard in order to represent the solid by using the approximating rectangular boxes (Figure 4). During sketching and labelling the three-dimensional object became a diagram on the drawing plane by means of experimenting with two-dimensional graphs. The video data show that all students observed this representation carefully. When the students had finished their solution one of them made a very important question:

Student A But, … what is the relationship between solving this problem and the double integral concept? In the integral of a one variable function we had the area, and now we will be able to obtain the volume of the solid, is that it?

This question was the opportunity for the teacher to ask students to look back at the textbook and look for the definition of double integral (Figure 5). Using this definition, students returned the solution of Example 1 and evaluated the volume of the solid as it is shown on the students' procedures in Figure 6.

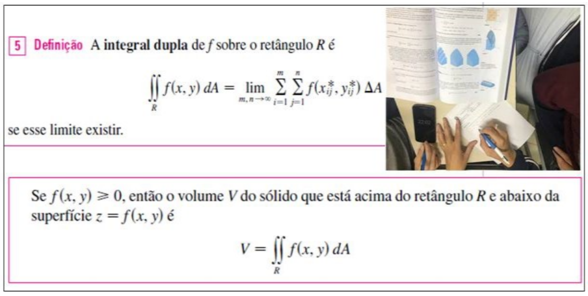


Figure 5. Double integral definition (Stewart, 2013, p. 876)

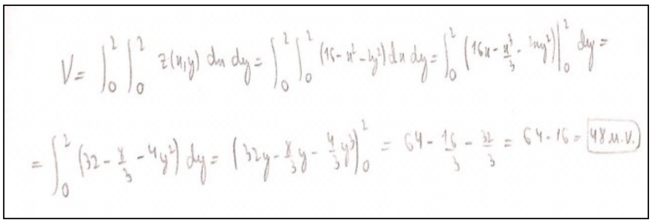


Figure 6. Obtaining volume using double integral definition

STUDENT ACTIVITY FROM A SEMIOTIC PERSPECTIVE

Peirce's characterization of diagrammatic reasoning may provide a theoretical frame for describing and interpreting the students learning activity and their use of diagrams in order to learn de double integral concept.

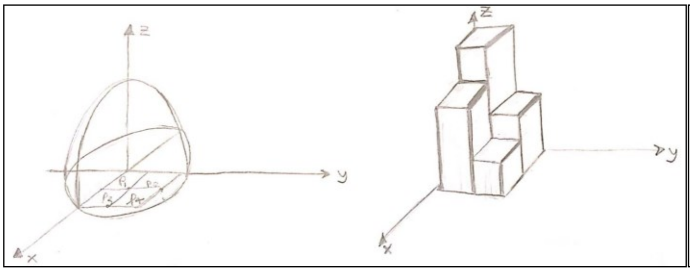
The preliminary study of students using the textbook had already given them opportunities for constructing and experimenting with diagrams. This construction and experimentation intensified when they solved the problem (Figure 2) without the use of the textbook.

The problem motivated the students to represent in the plane a diagram following certain rules. The first resolution (Figure 3) provided students the opportunity to experiment with diagrams. In fact, one of the students immediately made a graphical representation to explain his resolution to the other students (Figure 4). Others, however, discussed possibilities in an attempt to understand what they should do to solve the problem:

Student B: Let's have boxes under the paraboloid. The bases of the boxes are the squares that we have represented here [pointing to Figure 3]

Student C: Here on the plane we do the squares one by one. We have to draw four squares. Now from the upper left corner we consider the height of each box.

This dialogue encouraged student B to make a graphical representation - a diagram - to assist him in explaining his algebraic resolution. However, he was not yet able to made a representation in which the graph of the function and the approximating boxes were merged, but he put the two representations in different images (Figure 7). Another student (student C) was intended to explain the meaning of the volume he obtained by using the definition of double integral (Figure 6). With this purpose he made a graphical representation in which he tried to show the geometric meaning of the value he has got (Figure 8).



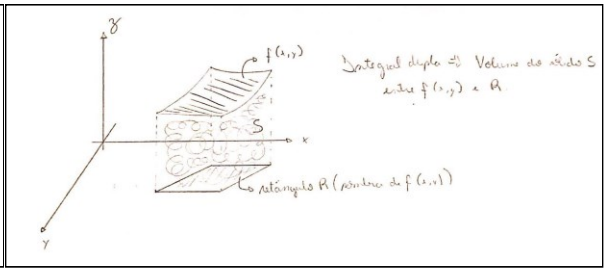


Figure 7. Diagrams constructed by a student

Figure 8. Student's diagram

The question the student addressed to the teacher (And what is the relationship between solving this problem and the double integral concept?) denotes that the preliminary study using the textbook (Figure 1) and the problem solving (Figure 2) enabled students to construct and make experiments with diagrams and that new relations between the different diagrams could be noticed by the students. The students' representation as presented on Figure 4 seems to be a clue of the possibility that the construction of diagrams allows perceiving relationships between the parts of a diagram other than those which are used in its construction.

The study of the double integral concept using the textbook led the students to make transformations in diagrams they had constructed and then they were able to evaluate the volume of the solid below the surface. The results that we can highlight in this activity lead us to state that the way the textbook introduces the double integral concept favors the students learning process. The students' procedures seem to have been supported by the diagrammatic reasoning triggered by images, problems, and language as included in the textbook.

References

Almeida, L. M. W., & Silva, K. A. P. (2018). A semiotic interpretation of the derivative concept in a textbook. *ZDM Mathematics Education. 50*(5), 881–892.

Hoffmann, M. H. G. (2005). Signs as means for discoveries. In M. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign: grounding mathematics education* (pp. 45–56). Boston, MA: Springer.

Kadunz, G. (2016). Diagrams as means for learning. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics* (pp. 111–126). Rotterdam: Sense.

O'Halloran, K. (2018). A Multimodal approach for theorizing and analyzing mathematics textbooks. In G. Schubring, L. Fan, & V. Geraldo (Eds.), *Proceedings of the Second International Conference on Mathematics Textbook Research and Development* (pp. 26–48). Rio de Janeiro: Instituto de Matemática, Universidade Federal do Rio de Janeiro.

Stewart, J. (2013). *Cálculo, volume 2.* São Paulo: Cengage Learning.

REPRESENTATION OF VECTORS IN GERMAN MATHEMATICS AND PHYSICS TEXTBOOKS

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The concept of vector is a central part of mathematics and physics at school. Three approaches to the concept of vector can be distinguished: arrow classes, n-tuples and vector space axioms. In order to develop adequate conceptions of vectors, different facets of the concept should be presented to the students and the representation in the subjects mathematics and physics should be coordinated. The method of textbook analysis was chosen to investigate this relationship with the help of a deductively developed system of categories. These categories are based on the theoretical framework using qualitative content analysis according to Mayring (2000). The following question was focused: Do German mathematics and physics textbooks include multiple representations of vectors in the introductory chapters?

INTRODUCTION

The concept of vector and vector calculus are central issues of modern mathematics as well as the subject of school mathematics in upper secondary schools. Filler & Todorova (2012) also mention this:

> "The vector concept belongs to the central structural concepts of mathematics and has manifold applications." (S.47, translated)

Indeed, according to Wittmann (1996), the decisive importance of the vector concept is achieved by its application in physics:

> "However, the vector concept gains its essential legitimation above all from its interdisciplinary relation to physics teaching. [...] This connection to the subject of physics [must] be sought and emphasized." (S.97)

Filler (2011) emphasizes the historical development of the vector concept from both geometrical and physical requirements:

> "The vector calculation was developed in a long historical process, mainly due to the need for a geometric calculation and the requirements of physics". (S.85)

Therefore, the handling of vectors is important in mathematics as well as in physics class. Dilling (2019) shows the possibility and benefit of an exchange of the research on vectors in mathematics and physics education for researchers as well as for teachers. In order to ensure an adequate development of the concept of vectors in school, students should get to know many different facets of the use of vectors. Furthermore, the presentations in the subjects of mathematics and physics should also be coordinated with each other. This article aims to clarify this relationship by examining the following research question:

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 155–160). Paderborn: Universitätsbibliothek Paderborn.

Do German mathematics and physics textbooks include multiple representations of vectors in the introductory chapters?

THEORETICAL FRAMEWORK

Different approaches to the concept of vectors are discussed in mathematics education research. Filler & Todorova (2012) distinguish the approaches via classes of arrows, n-tuples and vector space axioms.

When using arrow classes, vectors are defined as arrows of the same length and direction. The geometrical approach is of great importance for the students' understanding of the term. However, the identification of vectors with single arrows or concrete objects can entail difficulties. This problem is further intensified by the use of so-called location vectors. They are introduced to avoid the implementation of an addition of points and vectors and represent concrete vectors starting from the origin (Henn & Filler, 2015).

Malle (2005) strongly criticises the approach via arrow classes. Arrow classes are "useless" for analytical geometry, since students normally think in points and arrows, but not in arrow classes, as well as "useless" for physics, since almost all vectors occurring in physics are not arrow classes.

The access via n-tuples starts with vectors as n-tuples of real numbers. These are subsequently used for geometrical applications. This arithmetical approach often leads to a displacement of arithmetical perceptions by geometrical perceptions in the long-term development. Bender (1994) criticizes this approach for the role of linear algebra as a "trivializer". Geometric aspects are particularly important for the development of sustainable basic concepts. Therefore, the transition between an arithmetic and a geometric vector model should be practiced in class. Malle (2005) recommends introducing vectors as n-tuples and then interpreting them geometrically as points and arrows, but not as arrow classes.

A third possible access to the concept of vectors is the approach via vector space axioms. It was especially used in schools in the 1970s and is nowadays primarily part of linear algebra courses at university. Filler & Todorova (2012) describe this approach as technically elegant but nevertheless unsuitable for school education due to its complexity.

METHODOLOGY

In the following, the introductory chapters to vector calculation or kinematics of four mathematics and physics textbooks frequently used in North Rhine-Westphalia were selected to answer the research question: *Do German mathematics and physics textbooks include multiple representations of vectors in the introductory chapters?*

These are the following books:

- Brandt, D. et al. (2014). *Lambacher Schweizer Mathematik. Einführungsphase. Nordrhein-Westfalen.* Stuttgart: Klett (pp. 116–118).

- Grehn, J. et al. (2007). *Metzler Physik.* Braunschweig: Bildungshaus Schulbuchverlage (pp. 12–13).
- Krysmalsky, M. et al. (2014). *Fokus Mathematik. Einführungsphase. Nordrhein-Westfalen.* Berlin: Cornelsen (pp. 176–178).
- Meyer, L. et al. (2003). *Duden Physik. Gymnasiale Oberstufe.* Berlin: Duden Paetec (pp. 58–60).

The analysis of the textbook chapters is based on the rules of qualitative content analysis (Mayring, 2000). According to the deductive category formation method, a system of categories was established on the basis of theoretical foundations and precedes the analysis of the data material. A so-called coding agenda contains the definition of a category, an example derived from the data material and coding rules which, in the event of uncertainties, cause a clear classification of the categories. Based on the theoretical considerations, the following coding agenda was set (Table 1). The definition was sufficient for the assignment of the examples, so coding rules were not necessary to define. The whole material was coded by the author.

Category	Definitions	Examples
C1: Geometrical Approach	Vectors are introduced as arrow classes	Descriptive Text in *Lambacher Schweizer Mathematik* (p. 116): "In geometry, a vector can be described by a set of parallel arrows of the same length and orientation." (Translation from German language)
C2: Arithmetical Approach	Vectors are introduced as n-tuples	Definition 7.1 in *Fokus Mathematik* (p. 177): "A triple of numbers $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is called a vector." (Translation from German language)
C3: Axiomatical Approach	Vectors are introduced with vector space axioms	no example in the data material

Table 1. Coding agenda for the analysis of the four textbooks

RESULTS

The introduction of vectors differs significantly in the analysed textbooks.

In *Lambacher Schweizer Mathematik* vectors are introduced in the context of translations. In a coordinate system five arrows are shown, which all represent the same translation. This translation is described by a vector which indicates the translation in direction of x_1 and x_2. A vector is then defined as follows:

> "In geometry, a vector can be described by a set of parallel arrows of the same length and orientation. Such a set of arrows is already defined if one knows one of its arrows, a representative." (p. 116, translated)

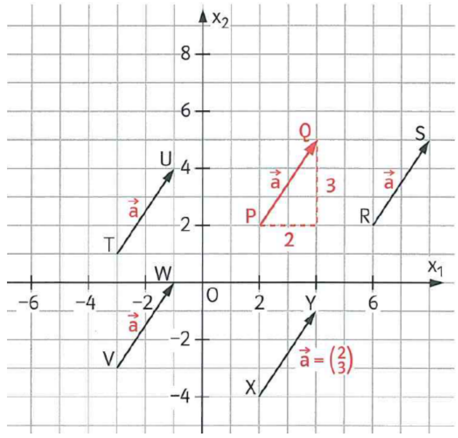

Figure 1. Excerpt from *Lambacher Schweizer Mathematik*

This introduction of the concept of vector can be assigned to the geometrical approach using arrow classes. The arithmetical and the axiomatical approach are not used in this textbook.

The book *Fokus Mathematik* uses the path between two points as the context for the introduction of the concept of vector. In a three-dimensional coordinate system, a fourth point C' is to be found to three points A, B and D, so that a parallelogram is created. To do this, the arrow between the two points A and D is determined. The fourth point C' is then determined by adding the same arrow to the point B. The concept of vector is then described as follows:

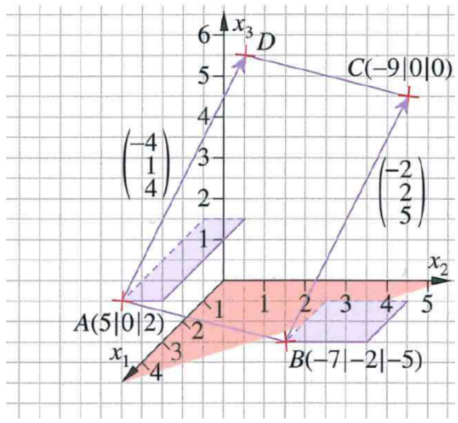

Figure 2. Excerpt from *Fokus* Mathematik

> "The two arrows are in different places, but they are both described by $\begin{pmatrix}-4\\1\\4\end{pmatrix}$. This common of both arrows is called a vector and each of the arrows $\overrightarrow{AD}$ and $\overrightarrow{BC}$ is an example or representative of this vector." (p. 176, translated)

After the description, the concept is defined separately on the next page:

> "A triple of numbers $\vec{v} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ is called vector. All arrows representing this vector $\vec{v}$ have the same length, are parallel to each other and have the same direction." (p. 177, translated)

The introduction of vectors is initially motivated geometrically on the basis of arrow classes. In addition to that, the definition is arithmetical and refers to triples of numbers, which are then interpreted as arrow classes.

The physics textbooks *Metzler Physik* and *Duden Physik* introduce vectors in the context of motion. The first vectorial quantity that is introduced is the path $\vec{s}$. In *Metzler Physik* the term is distinguished from scalar quantities as follows:

> "Since not only the magnitude of the path but also its direction is important for a change of location, the physical quantity of the path is a vector $\vec{s}$." (p. 13, translated)

In *Duden Physik* it says:

> "The path is a vectorial quantity, thus it is marked by magnitude and direction." (p. 58, translated)

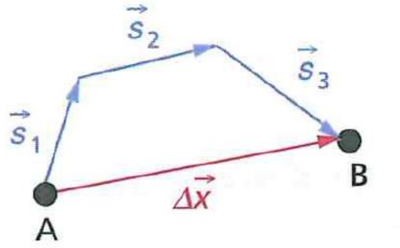

Figure 3. Excerpt from *Duden* Physik

The use of vectors to describe paths is related to the geometric approach to the concept of vectors. However, the equivalence of vectors with arrow classes is not mentioned. Such properties of the concept of vectors can only be recognized implicitly from the nature of movements. An arithmetical representation of vectors as tuples of numbers cannot be found in both works.

CONCLUSION AND OUTLOOK

In summary, it can be said that the geometrical approach predominates in both mathematics and physics textbooks. An arithmetical approach to the topic can only be found in one mathematics textbook; an axiomatical approach in none of the analysed books. In all textbooks there are many illustrations with vectors in the form of arrows; representations with n-tupels of real numbers can only be found in the mathematics textbooks.

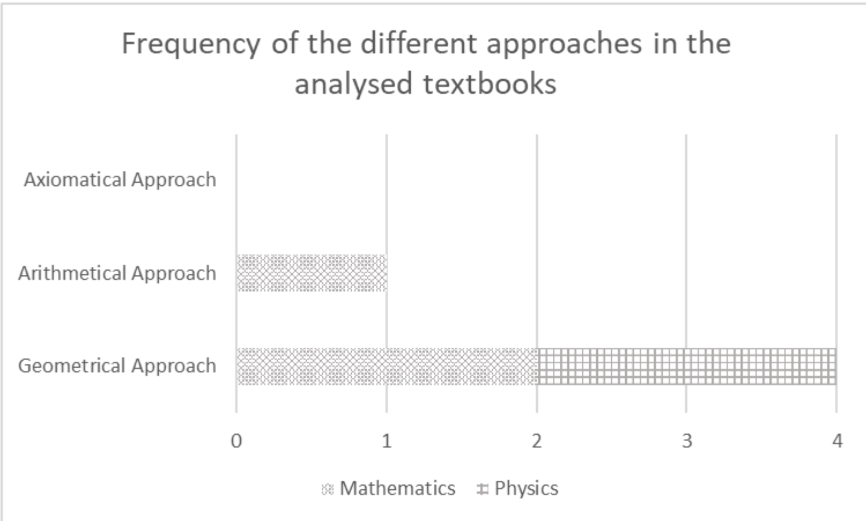

Figure 4. Comparison of the frequencies of the different approaches

The use of vectors in mathematics and physics classes should be coordinated so that students do not develop isolated concepts of vectors in the single subjects. For example, arithmetical representations of vectors should be used in physics lessons and physical applications should be discussed in mathematics lessons. To ensure the quality of such interdisciplinary approaches, teachers should not only learn the basics of the other subject but also the didactic basics of their studies. For such an exchange on didactical theoretical concepts, many different intersections between mathematics and physics education research exist. Such an interdisciplinary perspective on the research and the teaching level is being developed in various projects at the University of Siegen (cf. Holten & Krause, 2018; Krause et al., 2019).

References

Bender, P. (1994): Probleme mathematischer Begriffsbildung diskutiert am Beispiel der Vektor-Addition. *Mathematica Didactica, 17*(1), S. 3–27.

Brandt, D. et al. (2014). *Lambacher Schweizer Mathematik. Einführungsphase. Nordrhein-Westfalen.* Stuttgart: Klett.

Dilling, F. (2019). Vektoren in der mathematik- und physikdidaktischen Forschung. Stoffdidaktischverbindende Ansätze zwischen linearer Algebra und klassischer Mechanik. *Phydid B.*

Filler, A. (2011). *Elementare Lineare Algebra. Linearisieren und Koordinatisieren.* Heidelberg: Springer Spektrum.

Filler, A. & Todorova, A. D. (2012). Der Vektorbegriff. Verschiedene Wege zur Einführung. *Mathematik Lehren,* (174), 47–51.

Grehn, J. et al. (2007). *Metzler Physik.* Braunschweig: Bildungshaus Schulbuchverlage.

Henn, H.-W., & Filler, A. (2015). *Didaktik der Analytischen Geometrie und Linearen Algebra. Algebraisch verstehen – geometrisch veranschaulichen und anwenden.* Berlin, Heidelberg: Springer Spektrum.

Holten, K., & Krause, E. (2018). InForM PLUS vor der Praxisphase. Zwischenbericht eines interdisziplinären Elements in der Lehramtsausbildung an der Universität Siegen. *Proceedings of the Conference: Herausforderung Kohärenz, Praxisphasen in der universitären Lehrerbildung (HerKuLes).*

Krause, E., Bien, N. V., Chat, T. N., Chi, N. P., Dilling, F., Geppert, J., Holten, K., Le, T. A., Kraus, S., & Tho, C. C. (2019, in print). Inter TeTra – Interdisciplinary teacher training with mathematics and physics. Description of a project partnership between Siegen (Germany) and Hanoi National University of Education (Vietnam). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education.*

Krysmalsky, M. et al. (2014). *Fokus Mathematik. Einführungsphase. Nordrhein-Westfalen.* Berlin: Cornelsen.

Malle, G. (2005). Neue Wege in der Vektorgeometrie. *Mathematik Lehren,* (133), 8–14.

Mayring, P. (2000). Qualitative content analysis. *Forum: Qualitative Social Research*, *1*(2), Art. 20, http://nbn-resolving.de/urn:nbn:de:0114-fqs0002204.

Meyer, L. et al. (2003). *Physik. Gymnasiale Oberstufe.* Berlin: Duden Paetec.

Wittmann, G. (1996). Eine Unterrichtssequenz zum Vektorbegriff in der Sekundarstufe I. *Mathematica Didactica, 19*(1), 93–116.

MATHEMATICAL AND LINGUISTIC FEATURES OF WORD PROBLEMS IN GRADE 4 AND 5 GERMAN TEXTBOOKS — A COMPARATIVE CORPUS LINGUISTIC APPROACH

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Mathematical and linguistic features of word problems have been investigated with respect to their potential difficulties in various studies. In the current study, the transition from Grade 4 (primary school) to Grade 5 (German secondary school) is studied for identifying changing demands in these features. For this, the study focuses word problems using the basic rules of arithmetic in different German Grade 4 and Grade 5 textbooks. By a corpus linguistic approach, similarities among the features can be revealed as well as differences depending on the grade and the type of rule of arithmetic. The results of the study concerning these features and changing demands are relevant for designing teaching-learning arrangements, which prepare students to cope with typical features of word problems in Grade 5 textbooks.

BACKGROUND: MATHEMATICAL AND LINGUISTIC FEATURES OF WORD PROBLEMS AND THEIR POTENTIAL DIFFICULTIES

When dealing with mathematical word problems in textbooks, activities of the reader and features of the word problem text contribute to the reader's success. A specification of typical mathematical and linguistic features of word problems and their potential difficulties is hence required to decide which challenges student need to take when solving word problems. For other age groups, some of these features have been classified and investigated in former studies (cf. Prediger et al., 2018; Haag et al., 2013, Bergqvist et al., 2012). Those studies reveal potential difficulties in (German) word problems concerning the following features (cf. Dröse, in prep. for 2019):

- *Mathematical features:* The property of the included numbers and the type and number of operations are features that can be used to determine the complexity of a word problem (cf. Daroczy et al., 2015 for a review of the work of De Corte, Verschaffel and colleagues).
- *Factual features & factual-mathematical features:* The type and function of the question and context of a word problem (cf. e.g. van den Heuvel-Panhuizen, 2005) can influence the reader's understanding.
- *Semantic features & semantic-mathematical features:* The semantic structure (either static or dynamic) and the place of the unknown value can evoke difficulties (Riley & Greeno, 1988). Furthermore, the existence of irrelevant additional information can be demanding in the solution process (Muth, 1992).
- *Linguistic features* can be demanding e.g. on word, sentence, and text level (Prediger et al., 2018; Haag et al., 2014; Bergqvist et al., 2012):

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 161–166). Paderborn: Universitätsbibliothek Paderborn.

(1) The lexical complexity on the word level has been proven to be demanding especially when word problems contain flexion of words and complex word compositions (e.g. in the language of schooling).
(2) On the sentence level, the syntactic complexity being connected to the function of the phrases of the text can be demanding.
(3) On the text level, lexical-semantic features those feature that have an impact on the cohesion between sentences can evoke potential difficulties.

For further characterising the potential difficulty of linguistic features, their function for encoding and decoding the basic rules of arithmetic is decisive. The process of forming cohesion is influenced by the author encoding the basic rules of arithmetic and the reader decoding the basic rules of arithmetic. Therefore when describing potential difficulties, the linguistic-mathematic features of both processes on the three levels named above have to be taken into account. When considering those processes levels of encoding and decoding can be distinguished into the categories of technical term, verbal cue, interpretation on sentence level and interpretation on text level in ascending order of complexity. A systematic overview over those categories and their operationalization is presented in the following section (cf. Dröse, in prep. for 2019 for further explanation).

The features of word problems presented above and their potential difficulties have been investigated mostly in higher grades of secondary school or in lower grades of primary school. So far, there are no comparative studies focussing on possibly changing demands between primary school and secondary school.

RESEARCH QUESTIONS

In order to fill this research gap, the presented study focusses on word problems using the basic rules of arithmetic in German Grade 4 (primary school) and Grade 5 (secondary school) textbooks to inventory the presented features with potential difficulties and to investigate possibly changing demands. Therefore, a comparative corpus linguistic approach was used focussing on two research questions:

(Q1) Which mathematical, factual, semantic and linguistic features do word problems in German Grade 4 and Grade 5 textbooks contain?

(Q2) How do these features differ between grades and operations?

METHODOLOGY: COMPARATIVE CORPUS LINGUISTIC APPROACH

Composition of corpora

For the comparative corpus linguistic approach, the most often used Grade 4 and Grade 5 German textbooks of North Rhine Westphalia have been selected. For Grade 4, these textbooks are ‚Denken und Rechnen 4‘ (Version: 2012), ‚NussKnacker 4‘ (Version: 2010), ‚Welt der Zahl‘ (Version: 2011) and ‚Das Zahlenbuch‘ (Version: 2013). For Grade 5, these textbooks are ‚Schnittpunkt 5‘ (Version: 2006), ‚Mathewerkstatt 5‘ (Version: 2011/2012), ‚Mathe Live 5‘ (Version: 2006), ‚Zahlen und Größen 5‘ (Version: 2013), ‚Fokus Mathematik 5‘ (Version: 2013), ‚Lambacher Schweizer 5‘ (Version: 2005) and ‚Elemente der Mathematik‘ (Version: 2012).

25–30 word problems focussing on the basic rules of arithmetic were taken from each textbook. The *corpus of Grade 4 textbooks* (abbreviated G4) contains n=147 and the *corpus of Grade 5 textbooks* (abbreviated G5) contains n=289 word problems.

Procedure of the comparative corpus linguistic approach

Figure 1 shows the steps and categories of the inventory and analysis of the corpus linguistic approach, as well as their operationalization and a typical example.

Category for inventory	**Operationalization**	**Example**
1. Mathematical features		Lisa sticks 24 photos in a photo album, four photos on each page. For each photo, she needs four mounts. Tim sticks 36 photos in a photo album, six photos with three mounts on each page. Who needs more pages in the photo album? (Denken und Rechnen 4, p. 41 no. 3e)
Number of operations	Number of steps of applying the basic rules of arithmetic to answer the word problem's question.	
2. Semantic features & semantic-mathematical features		*Number of operation:* 3 steps 24 : 4 = 6 (Lisa needs 6 pages) 36 : 6 = 6 (Tim needs 6 pages) 6 – 6 = 0 (compare results of Lisa and Tim)
Additional information	Information given in the word problem that is not needed to answer the word problem's question.	
3. Linguistic features & linguistic-mathematical features		*Addition information:* four mounts, three mounts
A. Seriality of information	Order of information being consistent with the order of their processing to answer the word problem's question.	*Seriality of information:* Order of information consistent with order of operations

B. Categories of coding and the basic rules of arithmetic: (Structurally relevant phrases carrying the structure of the basic rules of arithmetic in the following categories)		**Example (for addition as basic rule of arithmetic)**	**4. Further inventory and analysis**
i. Technical term	Authors decoding the rule on word level in the technical term. Readers need to decode lexical features of this word.	e.g. add up *(German: addieren)*	(1) Inventory of lexical and syntactic features (2) listing typical features of language of schooling
ii. Verbal cue (action orientated)	Authors decoding the rule on word level by using verbal cues that describe the rules as actions. Readers need to decode lexical features and have to consider textual features.	e.g. put together *(German: zusammenfügen)* Tina has 4 marbles and Tom has 6 marbles. How many marbles do they have, when they **put** their marbles **together**?	
iii. Verbal cue (state orientated)	Authors decoding the rule on word level by using verbal cues that focus the rules' elements as states. Readers need to decode lexical and syntactic features and have to consider textual features.	e.g. altogether *(German insgesamt)* Tina has 4 marbles and Tom has 6 marbles. How many marbles do they have **altogether**?	
iv. Interpretation of one sentence	Authors decoding the rule on sentence level, by verbalising their elements context-related. Readers need to decode syntactical features of the sentence and have to consider textual features.	The journey starts at 6 o'clock and lasts three hours. When does the journey end?	Inventory of contexts differentiating between (1) semantic structure (2) used units of measurement
v. Interpretation of the whole text	Authors decoding the rule on text level, without using verbal cues or direct content-related verbalisation. For the readers no direct translation of words or phrases into the rules' elements is possible. The Readers need to consider textual features.	Two trains start their journey at the same time from the same central station in opposite directions. The first train drives 130 km/ h and the second train drives 110 km/ h. How much kilometre are they apart after one hour?	

Figure 1. Overview of categories for the corpus linguistic analysis

In all steps the inventory and analysis of phrases and features has been differentiated according to grade, textbook and type of basic rules of arithmetic.

SELECTED RESULTS

In order to answer research questions 1 and 2 selected results of the comparative linguistic approach are presented comparing grades and type of operation.

Mathematical, semantic, factual and linguistic features

The 'seriality of information', the existence of 'additional information' and the 'number of operations' in comparison of G4 and G5 are shown in Table 1.

Features	Seriality of information	Additional information	Number of operations				
	given / not given	given / not given	one	two	three	more	varying*
Corpus G4	77% / 22%	49% / 51%	43%	27%	13%	12%	5%
Corpus G5	76% / 23%	28% / 72%	34%	25%	16%	22%	4%

Table 1. Distribution of features in Corpus G4 / G5 (percentages of word problems)

*number of operation depends on the students' choice of steps as parts of the task require systematic variation

For Table 1, more than 3/4 of the word problems in G4 and G5 have a serial order of information, with no differences between textbooks in different grades. Concerning the existence of additional information, the grades differ significantly: Much more word problems in G4 have additional information than in G5. For the number of operations, the most striking difference is the reduction of word problems with one and the increase of more than three operations in comparison of G4 to G5.

Categories of encoding and decoding basic rules of arithmetic

For Table 2, all structurally relevant phrases carrying the information on the arithmetical structure were categorized with respect to the way the structure is encoded.

Operation/	Addition		Subtraction		Multiplication		Division	
Category	G4	G5	G4	G5	G4	G5	G4	G5
1.Technical term	0%	0%	0%	0%	0%	0%	0%	0%
2.Verbal cue (action)	0%	2%	5,0%	0%	3%	7%	6%	2%
3.Verbal cue (state)	36%	37%	30%	25%	16%	21%	6%	2%
4.Sentence level	19%	20%	10%	25%	26%	34%	52%	37%
5.Text level	44%	41%	55%	50%	55%	38%	36%	59%

Table 2. Distribution of categories of encoding and decoding the basic rules of arithmetic in Corpus G4 and G5 (percentages of word problems)

Table 2 reveals that most of the basic rules of arithmetic are encoded and decoded via the text level in G4 as well as G5. Differences occur between the rules of addition / subtraction, where the encoding and decoding via verbal cues (state) is the second most frequent category, and the rules of multiplication / division, where the encoding and decoding via the sentence level is the second most frequent category. Furthermore, changes in encoding and decoding the rules of arithmetic between G4 and G5 can be revealed: (1) concerning the rules of subtraction: There is an increase of encoding and decoding these rules via the sentence level. (2) Concerning the rules of multiplication and division: There is a decrease of encoding and decoding the multiplication via the text level, meanwhile (3) there is an increase of encoding and decoding division via the text level and a decrease of encoding and decoding it via sentence level.

Linguistic complexity features

Figure 2 shows the linguistic complexity features of the meaning related phrases that have been assigned to the first three categories of encoding and decoding.

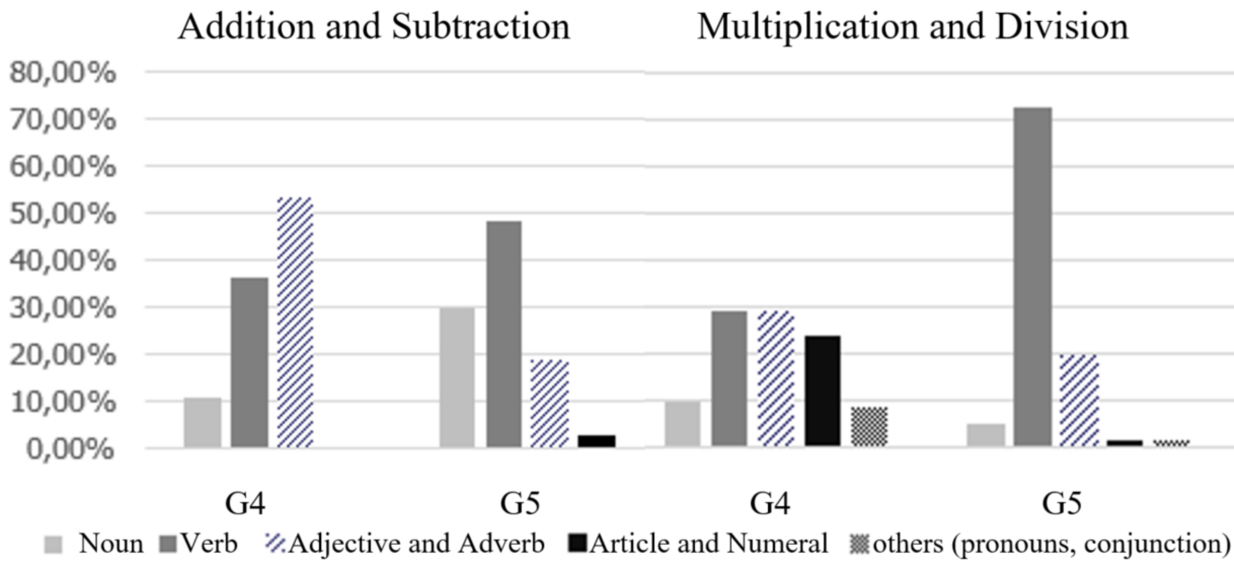

Figure 2. Linguistic features in structurally relevant phrases in Corpus of G4 and G5

It can be seen that there is a shift in the use of linguistic features between G4 and G5. In G4 adjectives / adverbs are the most frequent features, whereas in G5 verbs are the most frequent feature. A more detailed analysis reveals that comparatives only appear associated with addition / subtraction, while the derivation of adjectives only appear associated with multiplication / division.

> Example for comparatives: “(…) Tom jumps 24 cm more than Jens (*ger. weiter*). Jens manages 3,40 m. How fare does each of them jump?” (Nussknacker, p. 37 no. 1a)

> Example for derivation of adjectives: “Angela makes a cross-country run of 2600 m twice a week (German original *zweimal wöchentlich*). Which distance does she do in one year, when she trains 38 weeks a year?” (Mathe live 5, p. 59 no. 18)

CONCLUSION AND OUTLOOK

Some interesting findings on transitions in word problem features from G4 to G5 can be summarized: The mathematical features of ‘seriality of information’ occur rarely in G4 and G5, whereas ‘additional information’ is more often given in G4. In addition, word problems in G5 are more complex concerning the number of operations than in

G4. Therefore, teaching-learning arrangements should increase the number of operations in a controlled and not sudden way. Furthermore, for designing teaching-learning arrangements, the identified features of task should be taken into account, bearing in mind the described transition between G4 and G5, otherwise students might not be able to cope with unexpected word problem features (one approach is presented in Dröse, in prep. for 2019). Apart from the transitions between G4 and G5, the study indicates similarities and differences among the basic rules of arithmetic. For the encoding and decoding of all these rules, the text level is the most frequent category. That means a direct translation between word problem text and mathematical structure based on e.g. verbal cues on the word level might not be successful. This underlines the importance of understanding the deep structure of the text. Nevertheless further research on word problem features is needed especially for the transition of primary to secondary school.

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References

(Due to space limits, the references for the textbooks of the corpus are not listed here but can be requested from the author)

Bergqvist, E., Dyrvold, A., & Österholm, M. (2012). Relating vocabulary in mathematical tasks to aspects of reading and solving. In C. Bergsten, E. Jablonka, & M. Raman (Eds.), *Evaluation and comparison of mathematical achievement: Dimensions and: perspectivesProceedings of MADIF 8, The eighth mathematics education research seminar, Umea, January 24-25* (pp. 61–70). Linköping: SMDF.

Daroczy, G., Wolska, M., Meurers, W. D., & Nuerk, H.-C. (2015). Word problems: A review of linguistic and numerical factors contributing to their difficulty. *Frontiers in Psychology*, *6*, 348. doi.org/10.3389/fpsyg.2015.00348

Dröse, J. (in preparation for 2019). *Textaufgaben lesen und verstehen lernen*. PhD thesis in preparation, supervised by S. Prediger. TU Dortmund.

Haag, N., Heppt, B., Roppelt, A., & Stanat, P. (2015). Linguistic simplification of mathematics items: Effects for language minority students in Germany. *European Journal of Psychology of Education*, 30, 145–167. doi.org/10.1007/s10212-014-0233-6

Muth, K.D. (1992). Extraneous information and extra steps in arithmetic word problem. *Contemp. Educ. Psychol., 17*, 278–285. doi: 10.1016/0361476X(92)90066-90068

Prediger, S., Wilhelm, N., Büchter, A., Gürsoy, E., & Benholz, C. (2018). Language proficiency and mathematics achievement – Empirical study of language-induced obstacles in a high stakes test, the central exam ZP10. *Journal für Mathematik-Didaktik*, *39*(3). (Authorized English translation of Sprachkompetenz und Mathematikleistung. *Journal für Mathematik-Didaktik*, *36*(1), 77–104.) doi.org 10.1007/s13138-015-0074-0

Riley, M. S., & Greeno, J. (1988). Developmental analysis of understanding language about quantities and solving problems. *Cognition and Instruction*, *5*, 49–101.

Van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics*, *25*, 2–9.

TRANSITIONING FROM PRINT TO DIGITAL CURRICULUM MATERIALS: PROMOTING MATHEMATICAL ENGAGEMENT AND LEARNING

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The purpose of this paper is to report on research efforts to transition from a print problem-based curriculum to a digital environment. The goal of the project is to promote productive disciplinary engagement (Engle & Conant, 2002) in middle grades mathematics. In this paper, we highlight a collaboration feature of the digital environment that supports student engagement in mathematics. In doing so, we connect student collaboration in the digital platform with students' enacted experiences.

INTRODUCTION

Currently, too few students, including those in underrepresented groups, are engaged in making sense of mathematics (e.g., Organisation for Economic Co-operation and Development, 2013). As result, students are not empowered to use mathematics as a thinking tool to formulate, represent, and solve problems that they encounter in the world. And, in many classrooms, while it often appears to educators and scholars that students are engaged in problems in the classroom (e.g., the on-task behaviors and interactions such as actively speaking, listening, responding, and working), students may not be involved in the mathematical underpinnings of that engagement. While student engagement may be high, disciplinary engagement – student involvement in the concepts, practices, and issues of the discipline – is far too low in far too many places. In this paper, we report on design research efforts to promote productive disciplinary engagement (Engle & Conant, 2002) in a collaborative digital environment.

THEORY OF PRODUCTIVE DISCIPLINARY ENGAGEMENT

Students are productive in engaging in disciplinary practices when they make intellectual progress or demonstrate change in their conceptions over time related to the disciplinary learning goal (Hatano & Inagaki, 2003; Hiebert et al., 1996). This is known as productive disciplinary engagement (Engle & Conant, 2002). Examining student behaviors, participations, and interactions in classroom environments are essential for understanding the extent to which students are engaged in disciplinary practices with their peers (Williams-Candek & Smith, 2015). This is important because mathematics learning entails both communication and social relations (Sfard, 2008). The guiding assumption is that engagement in the activities of a discipline results in the learning of that discipline (Meyer, 2014).

If learning environments are to be effective, four design principles of productive disciplinary engagement need to be embodied: problematizing, authority,

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accountability, and resources (see Table 1). If all four principles are not embodied, then productive disciplinary engagement is not achieved (Engle & Conant, 2002).

Principle	Definition
Problematizing	Students address problematic situations that encourage uncertainties in mathematics (Hiebert et al., 1996; Engle & Adiredja, 2008; Zaslavsky, 2005).
Authority	Students share their thinking about a problem to become recognized as authors of the mathematics and contributors to the ideas of others (Lampert, 1990; Lehrer, Carpenter, Schauble, & Putz, 2000; Williams-Candek & Smith, 2015).
Accountability	Students take ownership of the mathematical ideas by making ongoing revisions to their work, communicating their ideas, and considering the reasonableness of the mathematics (Engle, 2011).
Resources	Students access a variety of resources such as time, location, technology, classroom artifacts to promote problematizing, authority, and accountability (Schoenfeld, 2012).

Table 1. Design principles of productive disciplinary engagement.

RESEARCH GOALS AND METHODS

The research reported in this paper is part of a larger design research study. One of the goals of the larger study is to promote productive disciplinary engagement (Engle & Contant, 2002). In this paper, we draw on conjecture mapping (Sandoval, 2014) to distinguish between the design of the digital materials, its development and the learners' enacted experiences within the classrooms. Specifically, we will report on how the design and development of the digital collaboration features of the environment can be connected with the four design principles of productive disciplinary engagement.

In our work, we are transitioning the problem-based middle-grades mathematics curriculum, *Connected Mathematics* (Lappan, Phillips, Fey, & Friel, 2014). In this curriculum, problems refer to contextual task situations that support some or all of the following: (a) has important, useful mathematics embedded in it, (b) promotes conceptual and procedural knowledge, (c) builds on and connects to other important mathematical ideas, (d) requires higher-level thinking, reasoning, and problem solving (e.g., mathematical practices), (e) provides multiple access points for students, (f) engages students and promotes classroom discourse, and (g) creates an opportunity for teachers to access student learning (e.g., formative assessment) (Lappan & Phillips, 2009).

Figure 1 shows a screenshot of the digital platform developed as part of the design research study. The features include support for real-time, synchronous collaboration among students with individual laptop computers. For example, in groups of 2–4, students can draw, make tables, generate graphs, upload photos, and write text on a workspace that are connected to others' workspaces. Students can publish their work and access other work that has been published. Students have access to a learning log record where they can document their understandings of big mathematical ideas that span beyond individual problems. For teachers, the digital platform includes features, such as (a) ability to access student group workspaces, (b) generate "just-in-time" prompts for individual students, groups, or entire classes, and (c) generate and publish work during whole-class discussions.

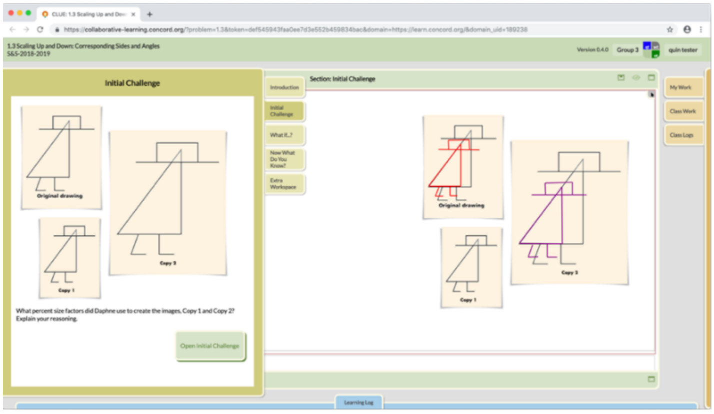

Figure 1. A screenshot of the student digital platform.

In the digital platform, mathematics problems were (re)designed in a new format, consistent with the focus on inquiry and investigation of mathematical ideas through rich problem situations, classroom discourse, and collaborations. In particular, the format of each mathematics problem foregrounds (a) the Initial Challenge, where students are exposed to the contextualization of the problem situation, (b) "What If...?" component, where students unpack the embedded mathematics, and (c) "Now What Do You Know?" where students connect learning to prior knowledge and consider future payoffs of the mathematics. Students can drag any workspace element embedded within the problem into the individual workspace and it becomes active.

The design of this digital platform both integrates and introduces features that are innovative in the ever-changing landscape of digital curriculum. For example, in Figure 1, students can move the figure on top of another, copy a part of the figure and compare it to a corresponding part of another figure. The utility of integrating problems with tools for mathematizing, such as a draw and graphing tool, in one integrated platform supports students' work efficiently and seamlessly. Further, the learning log feature which serves as a repository for artifacts generated throughout the problem-solving process provides an innovative way to support students' metacognition and self-reflection while also enhancing teachers' efforts to use formative assessment practices.

A MODEL FOR COLLABORATION IN THE DIGITAL ENVIRONMENT

One of the ways to promote productive disciplinary engagement in a group setting has been to focus on student collaboration. While students explore and solve the problem in their groups, they have collaboration supports so that digital work can be generated, shared, and accessed synchronously and in real-time by their group members. In the digital platform, students activate two mechanisms that support them in their collaboration. One mechanism is that students can allow other people to see their individual work in real-time. By giving permission, other students in their group can see every interaction performed in the workspace. This results in a video feed-like experience where every interaction is continually updated in real-time. To manage how students can see multiple members of their group's work in real-time, the second mechanism controls how students can see the work. When students activate this feature, their individual workspace transforms into a four-grid region where each region shows the individual workspace of the group. The upper left-hand region is their individual workspace, where the student can continually make changes to their work. In addition, students can also drag and drop copies of the digital work from their group members into their individual workspace and make further changes in real-time without re-creating the sequence of steps involved in constructing the object.

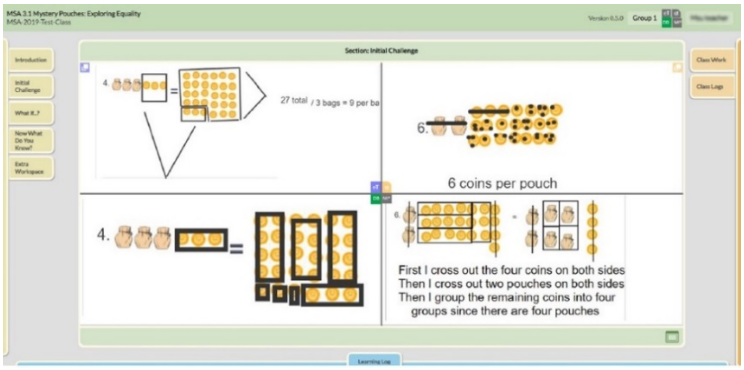

Figure 2. Collaboration among four students in the digital platform.

Figure 2 shows how a student can see how other students in the group interact with their work in real-time when they give permission. This model of collaboration affords students with ways into how others are thinking "in-the-moment" about the mathematics of the problem. This differs from paper-pencil environments where students cannot see others' written work in their group without interrupting their physical space, their thinking to ask questions, or their writing to see the work.

CONNECTING COLLABORATION TO STUDENT EXPERIENCES

The digital environment helps make visible student thinking in real-time during collaborative settings, which supports students' engagement in disciplinary practices. In terms of problematizing, the collaboration features support students to access the work and potential uncertainties that students document, including what students do, what they can conclude, how they justify, and what competing claims are generated. In terms of authority, the collaboration features support students to author their responses to the mathematics problem on their workspace, but also to share access of their response to others so that the group can recognize the contribution made by the students. The substance of the work reflects the various ways students are involved in the content, issues, and practices of mathematics in terms of formulating, representing, and solving problems. In classrooms using paper and pencil materials, the locus of control for access to others' thinking is often the teacher, including the teacher's work to establish norms for students working in groups. This digital environment implicitly supports authority by increasing students' control for sharing their work and accessing others' work. In terms of accountability, the collaboration features provide resources for students to be responsible for how their ideas make sense amongst the ideas of others. This could be from the perspective of how one's thinking compares to other problem-solving approaches, or it could be from the perspective of making sense of others' approaches and seeing how it relates to their own work. In a collaborative setting, students make ongoing revisions to their work, communicate their ideas, and consider how the ideas do or do not make sense so that they are better positioned to improve them when more thoroughly challenged. In terms of resources, the features underscore the flexibility and novelty of real-time, synchronous collaboration where students can easily share and access each other's work. And when building on a vast number of workspace resources, the features support a more dynamic way in which resources can be collaboratively used to explore and solve mathematics problems.

DISCUSSION

Providing students with opportunities to use collaboration features is important in inquiry-oriented, problem-based mathematics classrooms for several reasons. First, a student's capacity to make sense and represent his or her knowledge necessitates the ability to generate, critique, and refine their work. This is because student collaboration necessitates students to share their work and access other's work in real-time. In addition, since student work is publicly accessible and directly available to others through the collaboration features, the student work plays a vital role as students coordinate and navigate their mathematical understandings. The collaboration features enhance the ability for social interactions in classrooms to promote student problematizing, authority, and accountability in mathematics.

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References

Engle, R. A. (2011). The productive disciplinary engagement framework: Origins, key concepts, and developments. In D.Y. Dai (Ed.), *Design research on learning and thinking in educational settings: Enhancing intellectual growth and functioning* (pp. 161–200). London: Taylor & Francis.

Engle, R. A., & Adiredja, A. A. (2008). Increasing the cognitive demand of mathematical tasks. Paper presented at the Research Pre-Session of the National Council of Teachers of Mathematics conference, Salt Lake City, UT.

Engle, R. A., & Conant, F. R. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*, *20*(4), 399–483.

Hatano, G., & Inagaki, K. (2003). When is conceptual change intended? A cognitive-sociocultural view. In G. M. Sinatra & P. R. Pintrick (Eds.), *Intentional conceptual change* (pp. 407–427). Mahwah, NJ: Erlbaum.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12–21.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29–63.

Lappan, G., & Phillips, E. D. (2009). Challenges in US mathematics education through a curriculum development lens. *Journal of the International Society for Design and Development in Education*, *1*(3), 1–19.

Lappan, G., Phillips, E. D., Fey, J. T., & Friel, S. N. (2014). *Connected Mathematics 3*. Boston, MA: Pearson Student Edition and Teacher Edition.

Lehrer, R., Carpenter, S., Schauble, L., & Putz, A. (2000). Designing classrooms that support inquiry. In J. Minstrell & E. Van Zee (Eds.), *Inquiring into inquiry learning and teaching in science* (pp. 80–99). Reston, VA: American Association for the Advancement of Science.

Meyer, X. (2014). Productive disciplinary engagement as a recursive process: Initial engagement in a scientific investigation as a resource for deeper engagement in the scientific discipline. *International Journal of Educational Research*, *64*, 184–198.

Organisation for Economic Co-operation and Development (2013). PISA 2012 results in focus: What 15-year-olds know and what they can do with what they know. Paris: OECD.

Sandoval, W. (2014) Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences*, *23*, 18–36.

Schoenfeld, A. H. (2012). Problematizing the didactic triangle. *ZDM Mathematics Education*, *44*(5), 587–599.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.

Williams-Candek, M., & Smith, M. S. (2015). Principles of productive disciplinary engagement: Framework for design. Paper presented at the Research Pre-Session of the National Council of Teachers of Mathematics conference, Boston, MA.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, *60*(3), 297–321.

SAME TEXTBOOK, DIFFERENT POINTS OF VIEW: STUDENTS AND TEACHERS AS TEXTBOOK USERS

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The utilization of textbooks by mathematics teachers has been the subject of many recent studies; students' opinions, however, have not received such attention. The study presented in this paper aims to investigate both students' and their teacher's ways of and reasons for using the textbook, with an emphasis on the vertices of the Socio-Didactical Tetrahedron. The findings indicate that the beliefs about being a teacher and about being a student strongly influence textbook utilization. Also, the students' use of the textbook is influenced by the teacher's intentions. Here the extension of the didactical tetrahedron to a socio-didactical tetrahedron proved to be very valuable due to the social factors involved in textbook use.

INTRODUCTION

From the perspective of teaching and learning, the mathematics textbook is a very valuable resource designed for use by both students and teachers (Pepin, Gueudet, & Trouche, 2013). Many studies investigated teachers' use of textbooks (e.g., Pepin & Haggarty, 2001; Jukić Matić & Glasnović Gracin, 2016) and showed that textbook content and structure affect teachers' choices in different ways during the processes of both planning and enacting a lesson. One of those choices is the decision on the extent of textbook use in the classroom, which is reached by reflection on the textbook content and the aims of the teachers and those of the curriculum. The utilization of textbooks from the students' point of view has not received the same attention as the teachers' textbook utilization (Rezat, 2012). As a result of their teachers' intentions and choices, students mainly use textbooks as a source of exercises and for homework (Fan, Zhu, Qiu, & Hu, 2004). Still, studies that examine teachers and their students *together* in terms of textbook use are scarce. Such research has been conducted by Rezat (2012) and Viholainen, Partanen, Piiroinen, Asikainen, and Hirvonen (2015). These studies highlighted the need for further research in which teachers and students as textbook users are taken into account jointly, and not as separate research participants.

The study presented in this paper aims to investigate both the students' and their teacher's ways of and reasons for using the textbook, with an emphasis on the social aspects of their interaction with textbooks.

Social aspects of textbook use

Various social and institutional aspects, such as family, colleagues, peers, personal beliefs, rules and institutions influence the process of teaching and learning (Rezat &

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 173–178). Paderborn: Universitätsbibliothek Paderborn.

Sträßer, 2012), and thus may also affect how and why textbooks are used by teachers and students.

The aim of the study presented in this paper is to investigate and contrast the textbook utilization seen by a teacher and simultaneously seen by his/her students. For this purpose, we formulated the following research question: Which social and institutional parameters influence the teacher's textbook use and which parameters influence his/her students' use of the textbook?

THEORETICAL FRAMEWORK

The theoretical framework used to examine the utilization of mathematics textbooks from both the teacher's and students' point of view is Rezat and Sträßer's (2012) socio-didactical tetrahedron (SDT). This model derives from the original didactical tetrahedron with vertices: teacher, student, (mathematical) content and the artefact (textbook), which is extended by social and cultural influences positioned as the bottom vertices. These vertices and connected points are: public image of mathematics, norms about being a student/teacher, noosphere, institution, and peers and family. The model of the socio-didactical tetrahedron as proposed by the authors is presented in Figure 1.

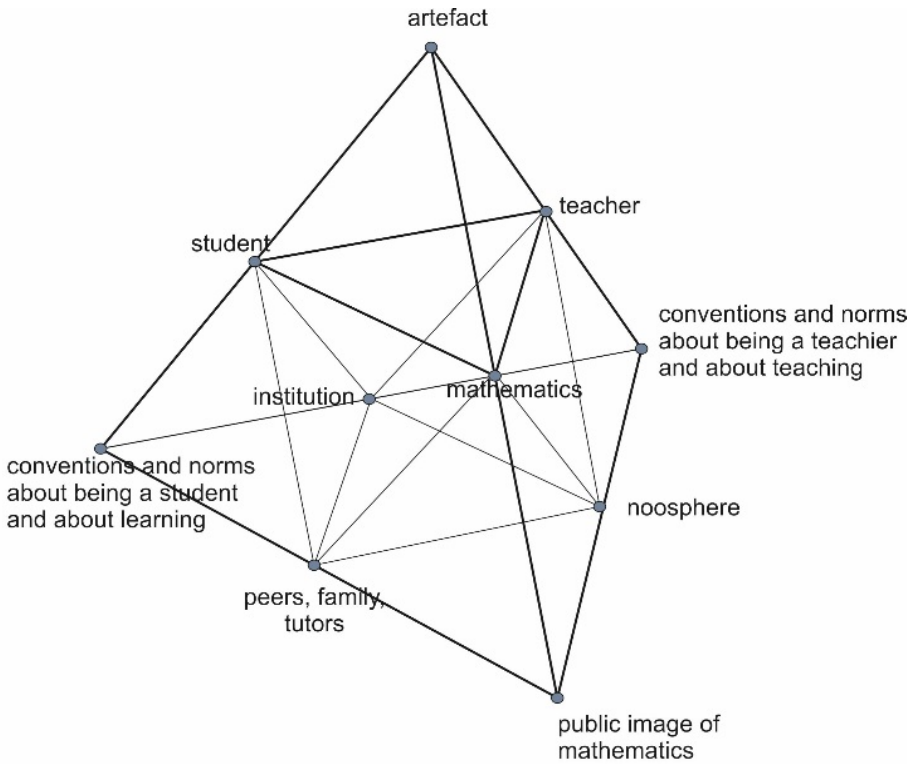

Figure 1. Socio-didactical tetrahedron (Rezat & Sträßer, 2012, p. 648)

The social and institutional parameters are often considered to be less visible because the persons involved often are not conscious of them. In this paper, we used the SDT model to find which connections among the vertices of the SDT become the most prominent and the most powerful for students and for their teacher during the process of using a textbook.

METHODOLOGY

The study presented in this paper is a case study which involved one female mathematics teacher from lower secondary education in Croatia (grades five to eight) and six of her students, divided into two targeted groups. The aim was the in-depth examination of participants' views and experiences on textbook utilization according to the research question. The criteria for teacher selection were: teaching experience (15 years), participation in previous studies and the utilization of textbooks in classroom practice. All the student participants were eighth-graders (age 14) and the teacher had been teaching them mathematics for the past four years (from grade 5 to grade 8). Students were chosen to focus groups according to their interest in the study and various school achievement.

The study encompassed the qualitative methods in the form of classroom observations (4 lessons), a semi-structured interview with the teacher, and interviews with each targeted group of students. Combining the observations and interviews provided a more in-depth understanding of the issues researched. Selected data presented in results reflect the compatibilities and divergences of teacher's and students' answers related to the research question.

RESULTS

The classroom observations showed that the textbook was used to a large extent for acquiring new content, for practising and for assigning homework. The interview showed that the teacher's beliefs and intentions affected the classroom enactment and the textbook use. In this paper the focus is on the two faces of the SDT model which contain the point Student. These are the tetrahedron faces *Textbook – Student – Teacher* and *Textbook – Student – Mathematics* (Figure 1).

The SDT face: Textbook – Student – Teacher

The participant students are asked to describe how they use the mathematics textbook. They said that they use it a lot, which is confirmed in the observed lessons. Furthermore, the students said that they use the textbook at home for homework, but also to prepare for the following lesson the next day:

> Student1: I use it [the textbook] when I get back home. If we have mathematics the following day, I read through some definitions in the textbook, go through several tasks and do the homework. Usually we have enough homework to go over and understand the content... and before a test I go through the definitions and I practice from the revision section in the textbook...

> Student3: I use it for homework, I use worked examples if I don't understand something and for revising before a test.

These responses refer to the SDT vertex *norms about being a student* which is connected to textbook use. For these participants, norms about being a student means: to use the textbook for homework, but also to prepare for the next lesson at home using the textbook and to revise for tests. In the interview, the teacher said that the textbook they use completely covers the content in the tests and that "if students use the textbook consistently they will achieve good results".

> Teacher: All the exam questions correspond to the tasks that we did in school or the students had them for homework.

The teacher believes that her job is to provide the content and tasks; her students should use the textbook both in the classroom and at home. In this way, the norms about being a teacher is connected to the students' utilization of the textbook and to the norms about being a student.

Regarding the relation Textbook-Student-Teacher, many details observed in the classroom correspond to the teacher's and students' beliefs from the interview. Decisions about whether or not to use the textbook, and what will be used, are made by the teacher in the preparation phase when she consults the textbook and other resources:

> Teacher: My students copy the definitions from the textbook, we also do the worked examples. But the motivational parts and acquiring new knowledge... sometimes we use the textbook for that, sometimes not. It depends on whether it is well presented in the textbook.

These teacher's decisions about the students' textbook use, are in line with the teacher's beliefs as a *mediator* between the textbook and students. This is also evidenced in classroom observations.

The participating students stated that from their perspective:

> Student2: She [the teacher] explains in her own words to make it clearer for us. The same content can be found in the textbook, but it's a little bit more difficult to understand.

> Student1: And then, she adds some interesting facts or something that would make it easier to learn... things that are not necessarily in the textbook.

The students were aware of the role of the teacher as a mediator between the textbook and the students, which is also a part of the norms about being a teacher from the SDT. The teacher claimed that she uses the textbook to introduce new content, but since she considers that the tasks in the textbook they use are not challenging enough, she also uses other professional resources. Here we notice the role of noosphere from the SDT model.

Although the teacher in the interview indicated that the textbook is used at home mostly by the stronger students, the results show that a weaker student uses the textbook as well, but for his own reasons:

> Student2: I do not use my notebook for studying, I always only use the textbook. In every subject. (...) Because I have very untidy handwriting. I use the notebook as a tool in school because it is obligatory; if I don't write things down in it I'll get a minus or an 'F'.

This response relates to another important point on the tetrahedron – institution, which is placed on the edge containing norms about being a student and norms about being a teacher. In Croatian schools the use of notebooks is a very important element of learning mathematics. This student, however, learns exclusively from the textbook at home because of his handwriting. He acknowledges that for institutional reasons he must have a notebook in school, but he uses the textbook for studying because it provides clear, structured and legible content.

SDT face: Textbook – Student – Mathematics

The teacher mentioned another reason why she relies on the textbook structure and content so much during her lessons:

> Teacher: I mostly think... that if someone did not understand something during the lesson, he or she can open the textbook at home and easily read through it.

The students think in a similar way:

> Student2: I really like that the textbook has worked examples; if something wasn't clear to me in school, I just look at the examples... I find the explanation in the book and after that I can solve the task by myself.

> Student3: If I wasn't present [at school], I just take a look in the textbook.

As well as being a help to students at home, the textbook may help parents and tutors. Here we come to the side point peers and family from the bottom (social) part of the SDT:

> Student2: I... I often get stuck because I have problems with mathematics, and when something is not clear to me, my father mainly helps me. (...) He uses the Internet and the textbook...

DISCUSSION AND CONCLUSIONS

The case study reported in this paper portrayed teaching and textbook use as social processes from two mutually dependent perspectives: the students' and the teacher's. The results show that the students' use of the textbook is dependent on the teacher-mediation process between student and textbook (e.g., Pepin & Haggarty, 2001; Love & Pimm, 1996). Further, many of the teacher's decisions concerning use of the textbook are influenced by the social component norms about being a teacher. This SDT vertex encompasses the teacher's pedagogical content knowledge as a strong component (An, Kulm, & Wu, 2004).

The teacher's and the students' answers in the interviews mainly correspond with the observed lessons (Jukić Matić & Glasnović Gracin, 2016). Also, the study showed the strong relation between the norms about being a teacher and the norms about being a student. Still, some hidden details and social and institutional reasons for textbook utilization were found in the students' interviews which were not evident during the observations and which the teacher was not aware of. One of these was the fact that a student uses the textbook a lot at home because his bad handwriting makes his notebook unusable.

Institutional reasons were also very important for understanding why the textbook was used so much in the school and at home. Namely, school as an institution has its rules, and textbooks in Croatia are obligatory and regulated by the Ministry of Education.

Many teachers and the public often believe that the use of textbooks depends mostly on the content (Jukić Matić & Glasnović Gracin, 2016), but this qualitative triangulation has shown another perspective. In addition to the content, teachers and students use / do not use a textbook for reasons related to institutions, conventions and norms of being a teacher and student, and for other social and institutional reasons. Therefore, it is useful to use the SDT in order to better understand the teaching and learning of mathematics.

References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7*, 145–172.

Fan, L., Zhu, Y., Qiu, X., & Hu, J. (2004). Textbook use within and beyond mathematics classrooms: A study of 12 secondary schools in Kunming and Fuzhou of China. In F. Lianghuo, W. Ngai-Ying, C. Jinfa, & L. Shiqi (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 228–261). Singapore: World Scientific.

Jukić Matić, L., & Glasnović Gracin, D. (2016). The use of the textbook as an artefact in the classroom: A case study in the light of a socio-didactical tetrahedron. *Journal für Mathematik-Didaktik, 37*(2), 349–374.

Love, E., & Pimm, D. (1996). "This is so": A text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education.* (Vol. 1, pp. 371–409). Dordrecht: Kluwer.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: a way to understand teaching and learning cultures. *ZDM Mathematics Education, 33*(5), 158–175.

Pepin, B., Gueudet, G., & Trouche, L. (2013). Re-sourcing teacher work and interaction: New perspectives on resource design, use and teacher collaboration. *ZDM Mathematics Education, 45*(7), 929–943.

Rezat, S. (2012). Interactions of teachers' and students' use of mathematics textbooks. In G. Gueudet, B. Pepin, & L. Trouche (Eds.)*, From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 231–246). New York: Springer.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM Mathematics Education, 44*(5), 641–651.

Viholainen, A., Partanen, M., Piiroinen, J., Asikainen, M., & Hirvonen, P. E. (2015). The role of textbooks in Finnish upper secondary school mathematics: Theory, examples and exercises. *Nordic Studies in Mathematics Education, 20*(3–4), 157–178.

MEANING MAKING SCHOOLBOOK MATERIALS FOR PROMOTING SUBTRACTION WITH REGROUPING SKILLS

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In Germany, primary school children should learn to not only execute the subtraction algorithm, but also understand why the algorithm works. Thus, the German textbook "Das Zahlenbuch" ("The Number Book", Nührenbörger et al., 2018) has designed language sensitive support material based on meaning-related vocabulary for understanding the subtraction algorithm. This paper gives an insight in the individual learning processes of a fourth grader who was fostered with this support material.

LEARNING THE SUBTRACTION ALGORITHM

In their third year of school, primary school children should learn how to solve subtraction tasks by using a subtraction algorithm. Which algorithm the children learn is in many states of Germany an individual decision of the mathematics teacher and in consequence a challenge for schoolbook authors. Thus, many schoolbooks offer materials and tasks for different (mostly two) subtraction algorithms, so that the teachers can decide which subtraction algorithm they want to teach. The most commonly used subtraction algorithms (Selter et al., 2012) are the following:

Equal addition: This algorithm is based upon the idea that adding the same amount to two numbers will not affect the difference between these two numbers. So, ten units are added to the minuend and to equal this addition one unit of the next higher place value is added to the subtrahend. This algorithm can be conducted by using the taking away or determining the difference strategy.

Austrian subtraction: This method takes advantage of the fact that subtraction is the inverse operation of addition. The core idea is to find out which number must column-wise be added to the subtrahend to obtain the corresponding number in the minuend.

Regrouping: This subtraction algorithm is also known as decomposition. By manipulating the minuend one group of a column must be regrouped in ten smaller units of the next smaller column. The decomposition method is used in many countries.

Pupils' errors in written subtraction are systematically evaluated in the last decades (e.g., Cox, 1974; Gerster, 1982). But less is known about how children can learn to understand the algorithm. Especially for the regrouping algorithm Brownell and Moser (1949) found out that this algorithm leads to many systematic errors if this method is primarily introduced in a formal way. Jensen and Gasteiger (2019) discovered that only five of 113 children in total could explain how the algorithm with regrouping works, whereas 108 of these fourth graders proceed entirely mechanically, at most using and describing basic facts without any understanding.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 179–184). Paderborn: Universitätsbibliothek Paderborn.

Mosel-Göbel (1988) compared the understanding of different methods (regrouping, equal addition and two variations of the Austrian subtraction) in four third classes (each class had a different method). She found no relevant differences with respect to success rates. But especially in one class the children were able to explain the algorithm. Since the analyses are based on videotaped data it could be worked out that these children were used to describing the subtraction algorithm with regrouping by means of meaning-related vocabulary: "A ten must be changed into ten units. Now the tens must be decremented." This indicates that perhaps a special way of classroom discourse could help children to understand the algorithm: a discourse that focusses on understanding connections and relationships by using a meaning-related language in the first place.

Meaning-related in this context means that the words and phrases children used express the relevant connections and the core idea of a mathematical content (Prediger & Wessel, 2013). While fostering mathematical concepts like fractions (Prediger & Wessel, 2013), percentages (Pöhler & Prediger, 2015) or multiplicative concepts (Götze, 2019) under low-achieving learners it has already become apparent that this meaning-related vocabulary promotes conceptual understanding.

Thus, a learning arrangement must not only provide opportunities to activate students' individual resources, it must also provide words and phrases that grasp the relations and meanings (Pöhler & Prediger, 2015) and visualizations that illustrate these relations and meaning. Both core ideas (visualization and meaning-related vocabulary) are basic approaches for conceptualizing language-sensitive schoolbook material in "Das Zahlenbuch" ("The Number Book", Nührenbörger et al., 2018) for promoting the understanding of for example the subtraction algorithm with regrouping. Therefore, the research project described below focuses on the following research question:

How does language sensitive schoolbook material help children to understand the subtraction algorithm with regrouping?

LANGUAGE SENSITIVE SCHOOLBOOK MATERIAL

In the revised edition of the German primary schoolbook "Das Zahlenbuch" ("The Number Book", Nührenbörger et al., 2018) we do not focus on fostering routines and technical terms in the first place. Instead, we focus on meaning making illustrations that can help children understand the mathematical content. Therefore, the illustrations are designed with regard to the design principles:

1) scaffolding meaning making language and
2) illustrating the mathematical content by means of graphical representations like the number line or by using manipulatives like the base-ten blocks or iconic representations of them.

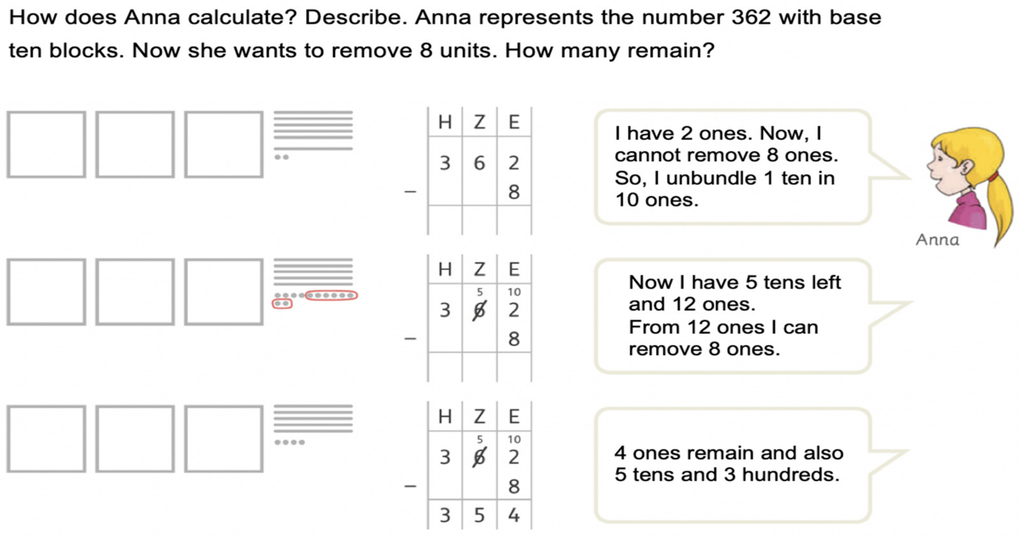

Figure 1. Introduction of the subtraction algorithm in "Das Zahlenbuch" ("The Number Book", Nührenbörger et al., 2018, translated by the author)

For the subtraction algorithm with regrouping an illustration was developed (see Figure 1, translated by the author) that represents the core idea of the algorithm. By implementing such illustrations in the schoolbook two aims are being pursued:

- On the one hand teachers get an idea of how to introduce and visualize the subtraction algorithm and how they can talk with the children about this mathematical content.
- On the other hand, the technical terms and especially phrases combined with the iconic representations of the base-ten blocks can function as a model for further (individual) subtraction with regrouping tasks. The children have a shared language to talk about how the algorithm works.

However, this language does not primarily address the technical function but the mathematical concept of the algorithm. Whether and to what extent this support material can help children to understand the subtraction algorithm with regrouping will be shown by means of the in-depth analysis of Osman.

THE INDIVIDUAL LEARNING TRAJECTORY OF OSMAN

Osman is a fourth grader with migration background. His parents are from the Arabic states. They have emigrated to Germany a couple of years ago. As Osman's first language is not German but Arabic his German language competences are not well-developed. Besides, his mathematical competences are rather below average. He often needs further support in mathematics class.

At the beginning of the support Osman should calculate some subtraction tasks on his own. In particular, he made many (typical) mistakes if a zero occurred in the minuend or if the subtrahend had fewer digits than the minuend. Furthermore, he had difficulties to explain the algorithm as it is shown in the following scene in which Osman explains how he has calculated 713-281.

Osman: I had just calculated from the top to the bottom. And now, you can subtract 1 from 3. And then, I have taken 1 from the 10. And then I have … no I mean, I have taken 1 from the hundreds. And then, I have deleted this one (points to the hundreds in the minuend) and have written a 6. And here (points to the tens in the minuend) I have written 11. And now you can take away 8 from 11, is 3. And here a 4 (points to the hundreds in the result).

In fact, Osman gave a description of what he has done. He knew that he has to cross out and reduce a number and to write a ten above. But in fact, he did not express understanding. He described what he has to do, but has not eplained why he could calculate in this way.

To support an understanding of the algorithm the teacher presented the schoolbook illustration (Figure 1). Osman had to find out the mathematical meaning of Annas explanation and calculations steps for the task 362-8.

Osman: Stop, ehm, these tens, this ten here they have made ones of this ten. And that is why they have 12 ones, now (counts silently). Correct, 12 ones.

Teacher: Exactly, And in the last step. What does Anna calculate?

Osman: 12 minus 8, results in 4. (Read it out loud) 4 ones remain and also 5 tens and 3 hundreds.

Teacher: Okay, to understand better what Anna means we can use the base-ten blocks. (Teacher gives Osman the base-ten blocks. Osman models 362.)

Osman: We want to calculate the task like Anna (looks at the schoolbook illustration). And now, we want to reduce this (takes a ten rod) and then build ones. And then we have just removed a ten rod. And now we remove 7 ones. 7 or 8? Oh no, 8. And now we have finished.

In this scene Osman primarily imitated Annas calculation steps and words. He always matched what he did and what Anna did in the schoolbook illustration. But while doing this, he is forced to link and combine the regrouping steps with his concrete actions and the formal written subtraction algorithm by using Annas words.

These concatenating words seem to be very important for interlinking illustrative examples expressing the mathematical content and the formal algorithm. This can be seen in the further development of Osman:

Osman: (calculates 212-8 with the base-ten blocks) So, 2 hundreds, 1 ten, 2 ones (models the number with material). Minus 8. Now I must reduce these tens (takes a ten rod), and I have to take 8 ones, oh no, 10 ones, I mean (takes 10 ones). Now, minus 8 (puts 8 ones aside). And the result is ... 200, 204.

Teacher: And what happens, if you write it as written subtraction?

Osman: 2 minus 8, does not work (writes a 10 in the unit-place and deletes the 1 in the tens-place of the minuend and writes a 1, then he calculates correctly). Ready.

Osman calculated correctly with the base-ten blocks as well as with the subtraction algorithm. Though, it would be too hypothetical to speculate that the support has led to initial success. Therefore, Osman's explanations do not indicate that he understood how the illustration with the base-ten blocks and the written algorithm are connected. He used the words of Anna only for working with the manipulatives but not for the written algorithm. At this point he has not realized that he could explain and think about the formal written algorithm in the same way as if he was calculating with the base-ten blocks. After the teacher has interlinked the way of thinking, the meaning-related vocabulary, and the written subtraction, Osman could explain the algorithm.

Osman: (calculates 652-256 with the subtraction algorithm without using base-ten blocks) 2 minus 6, that does not work. Therefore, I have to take a ten and unbundle it. Then I have 10 ones. 10 minus 6, oh no, 12 minus 6 are 6. (...) 4 minus 5, that does not work. Then I take 1 from the hundreds and make 10 tens. Then, there (points to the tens-place) I have a 4, and now I have a 5, no a 14. 14 minus 5 is 9 and 5 minus 2 is 3.

Osman could explain the core idea of the subtraction algorithm correctly. Therefore, he described the unbundling process in the tens and hundreds digits with the words and phrases of the base-ten blocks ("…take a ten and unbundle it. Then I have 10 ones.", "Then I take 1 from the hundreds and make 10 tens."). As he did not have the base-ten blocks nearby, it seems like Osman now realized that he could think about the formal algorithm in the same way as he had calculated previously with the base-ten blocks. Osman had successfully transferred this language.

DISCUSSION AND FINAL REMARKS

This short insight of the learning trajectory of Osman shows that the learning progress for understanding the subtraction algorithm with regrouping seems to depend on how deep the handling with the material is interconnected with the formal algorithm. It becomes obvious that meaning-related words and phrases could help to understand and concatenate concrete and symbolic representations. Therefore, the core idea of regrouping should be worked out with manipulatives linked with a meaning-related vocabulary that describes this concrete process (e.g. "I change/unbundle a ten to ten ones."). But after this – in a second step – this way of thinking and explaining must be transferred to the formal subtraction algorithm. Hence, the meaning-related vocabulary seems to have an intermediary function. Consequently, the language of thinking must be fostered more offensively as a language of the learners and not only as language of the teachers. Since in the concrete material processes the children could tip or link to something if they miss words—and most teachers are content with such deictic gestures—it is a big challenge and sometimes nearly impossible for the children to explain the formal algorithm.

For designing schoolbook material this study has shown that this meaning-related vocabulary should be implemented comprehensively in the schoolbook for interconnecting concrete or iconic representations with formal calculations and thus, for supporting understanding.

Nevertheless, implementing meaning-related vocabulary in schoolbooks cannot be taken as guarantee for learning progress. On the one hand, it highly depends on how the children internalize the concrete material handling and the meaning-related vocabulary as mental pictures and as thinking language. On the other hand, it depends on how the children are supported in realising that these mental pictures and this thinking language can help to explain formal calculations and algorithms. For this, meta communicative processes in classroom discourses become important. However, such meta-processes cannot be part of a schoolbook. But a schoolbook could lay the foundation for doing this.

References

Brownell, W. A., & Moser, H. E. (1949). *Meaningful versus mechanical learning: A study in grade III subtraction*. Durham: Duke University Press.

Cox, L. S. (1974). Analysis, classification, and frequency of systematic error computational patterns in the addition, subtraction, multiplication, and division vertical algorithms for grades 2-6 and special education classes. Kansas City: Kansas University

Gerster, H.-D. (1982). *Schülerfehler bei schriftlichen Rechenverfahren.* Freiburg: Herder.

Götze, D. (2019). *The importance of a meaning-related language for understanding multiplication* (Accepted paper). Utrecht: CERME 2019.

Jensen, S., & Gasteiger, H. (2019, in press). „Ergänzen mit Erweitern“ und „Abziehen mit Entbündeln“ – Eine explorative Studie zu spezifischen Fehlern und zum Verständnis des Algorithmus. *Journal für Mathematik-Didaktik*. https://doi.org/10.1007/s13138-018-00139-3

Mosel-Göbel, D. (1988). Algorithmusverständnis am Beispiel ausgewählter Verfahren der schriftlichen Subtraktion. *Sachunterricht und Mathematik in der Primarstufe, 12*, 554–559.

Nührenbörger, M., Schwarzkopf, R., Bischoff, M., Götze, D., & Heß, B. (2018). *Das Zahlenbuch 3.* Leipzig: Ernst Klett.

Pöhler, B., & Prediger, S. (2015): Intertwining lexical and conceptual learning trajectories: A design research study on dual macro-scaffolding towards percentages. *Eurasia Journal of Mathematics, Science and Technology Education, 11*(6), 1697–1722.

Prediger, S., & Wessel, L. (2013). Fostering German language learners' constructions of meanings for fractions: Design and effects of a language- and mathematics-integrated intervention. *Mathematics Education Research Journal, 25*(3), 435–456.

Selter, C., Prediger, S., Nührenbörger, M., & Hußman, S. (2012). Taking away and determining the difference: A longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79*(3), 389–408.

MATERIALS FOR INCLUSIVE MATHEMATICS EDUCATION — DESIGN PRINCIPLES AND PRACTICES

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In this paper, we aim to identify inclusive practises in maths lessons of inclusive primary schools. Based on the textbook "Das Zahlenbuch" and the accompanying artefacts, the lessons should offer occasions for all children to participate in and benefit from the learning situation. In our study, we analyse videographed lessons, considering the ideas of sensitivity of differences, language and materials and connections of content-related and social learning. In this paper, the results of the qualitative analysis of an introductory phase with all children in the field of simple subtraction tasks are discussed.

INTRODUCTION

Inclusive education addresses all children in their diversity and their different needs and abilities. "Inclusive schools aim to involve all learners in quality learning experiences which empower them to become active participants in a more equitable system" (Scherer et al., 2016, p. 640). Inclusive education focuses on the dismantling of barriers and on creating universal approaches of learning. With respect to mathematical lessons, inclusive education has to enable all children to develop basic mathematical competence in interaction with others (Häsel-Weide & Nührenbörger, 2017). Artefacts – like textbooks, workbooks, worksheets or visual aids – are an important part that constitute the inclusive maths lesson. Artefacts mediate students' classroom activity with mathematics, students interact with the artefact, the teacher and peers (Rezat & Sträßer, 2012). We ask how artefacts can be designed for inclusive education, and which inclusive practice can be reconstructed in lessons, when teachers use of the designed artefacts.

DESIGN PRINCIPLES FOR INCLUSIVE MATHEMATICS EDUCATION

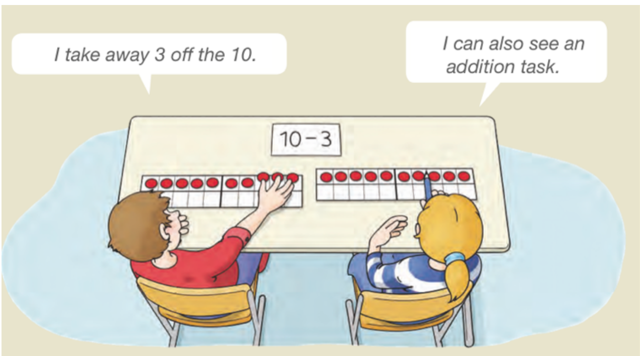
Inclusive mathematics education is asked to bring all children in contact with central mathematic contents. For mathematical learning, productive and interactive learning in substantial teaching units are seen as the main principles (Wittmann, 2001), especially. The instruction has to contain challenges for all children to widen their knowledge, to discover mathematic structures and to speak about mathematics. Some children need assistance to participate in the lessons, some need help to reach mathematic insight or to formulate their thoughts, but "all learners should be confronted with complex learning environments characterised by investigative learning and productive practicing." (Scherer et al., 2016, p. 641).

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 185–190). Paderborn: Universitätsbibliothek Paderborn.

The textbook “Das Zahlenbuch“ (Nührenbörger et al., 2017) is designed on the basis of active-discovering and socially-interactive processes in holistic mathematic structures, the central didactic principles of mathematical learning (Wittmann, 2001). In order to achieve this, the textbook works with learning environments and representations, which are substantially and spirally linked. This creates the possibility that each child is able to solve mathematical tasks on their own level. In this sense, inclusive teaching does not require a special form of teaching mathematics, but focuses on the individual strengths of each child and uses the heterogeneity of a learning group to learn in exchange with each other.

The handbook for promoting children with mathematical learning difficulties by working with the textbook “Das Zahlenbuch” (Häsel-Weide & Nührenbörger, 2017) highlights the idea of *differential sensitivity* i. e. differential, reflective perception of the heterogeneous competencies of a group in a concrete learning situation and an appropriate reaction. The handbooks helps teacher to use the artefacts of the “Zahlenbuch-Programm” and to modify the mathematical tasks adaptive to *different competences* and *individual needs* (Häsel-Weide & Nührenbörger, 2017). The influence of academic and content *language* for the understanding of mathematical concepts is highlighted as well as the use of appropriate material. Different learning situations are analysed with respect to their requirement of *social behaviour*, cooperative learning and potential of mathematical understanding.

Example:



Lay and calculate.

10 − 3 = ___	10 − 1 = ___	10 − 2 = ___	10 − 5 = ___	10 − ___ = ___
10 − 4 = ___	10 − 6 = ___	10 − 7 = ___	10 − 9 = ___	10 − ___ = ___

Figure 1. The illustration of the “10 - …” tasks in the textbook “Das Zahlenbuch 1” (Nührenbörger et al., 2017, p. 83)

The idea of subtraction can be interpreted as “taking away from” and “completing to” (Figure 1). To understand the concept of “taking away”, it is important that numbers are imagined as (structured) quantities. Children have to become aware of the differentiation of easy and difficult tasks. The simple tasks (e.g., “10 - …”, "halves", "minus 1" or "minus 5") should be used to derive the results of the difficult ones.

Children with mathematical learning difficulties have great issues in seeing the structure, understanding the operation and deriving results (Häsel-Weide & Nührenbörger, 2017). According to the idea of sensitivity of different competences, this must be considered in inclusive classes. To understand the concept of “taking away”, it is important to know as basic subject that numbers are imagined as (structured) quantities. Verbalising the subtractive action helps to imagine the shifting of the dots but a lot of phrases are needed to express the representation. The position of the dots in the twenty ten frame (Figure 1) allows to see the minuend, the subtrahend and the difference as well as the operation (moving the dots). The handbook describes presumable problems with the task and makes suggestions how they can be considered in different learning situations.

METHODS

The project is embedded in the research paradigm of didactic development research in the sense of Design Science (Nührenbörger et al., 2016). We are interested in designing artefacts and learning environments for inclusive education, and are analysing in detail which inclusive practices can be reconstructed. The analyse follows an interactionist perspective, focussing on the *classroom microculture* and *mathematical practices*. Mathematical practices are established as a "theoretical construct that allows us to talk explicitly about collective mathematical development" (Cobb, 1998, p. 34). They arise in an interactive process of classroom discourse.

In detail, we tackle three research questions in the study, but in the following, we focus on the first question.

- RQ1: How are plenaries in inclusive classes characterised regarding the idea of differential sensitivity?
- RQ2: How are peer-interactions characterised, in which students with different competencies are working in a common learning situation?
- RQ3: How do children with difficulties in math learning act when they work on adaptive tasks?

This study is part of two years of intensive joint work between the designers of the materials (both of the authors) and three teachers of two elementary schools. The cooperation is inspired by the idea of professional learning communities and collegial reflections. In the first year of the project, the children were in first grade, in the second year in the second grade. The lessons are videographed and used to a) answer the research questions as well as b) to reflect jointly the quality of the learning.

The data is analysed in different ways, according to the different questions. To answer the first research-question, we use the qualitative content analysis (Mayring, 2015). First, the different aspects of sensitivity are used as categories to code the introductory phase and the reflection period in a deductive way. The coding is taken on the digital videorecording and the found situations are described in a nutshell. Additionally, situations (which seem to be typical for inclusive math education but do not fall in a category) are marked and new categories are found in an inductive way.

ANALYSIS OF AN EPISODE: INCLUSIVE PRACTICES

In the following the introductory phase about "simple subtraction task" is exemplary analysed. Table 1 is showing the distribution of different inclusive practices during the introductory phase. All children participate in the plenum and are introduced to figure out subtraction task with minuend 10 on a twenty ten frame with dots (Figure 1). The task 10 – 2= 8 is solved exemplarily and the result is noticed on the work sheet.

	Subcategories	Concrete phrase	Description
Sensitivity of differences in competencies	Basic subject	"How much is it?" (teacher points at the ten frame) "Tell me, how can you see that there are ten dots without counting?" Finn: "If two boats are filled completely, because each boat contains five dots and then you are able to see immediately that it is ten". "Amalia, how much is five and five" (teacher shows two hands)?	Estimation of quantities based on structures
	Regular subject	"I tell you, what to do: I take away two dots from the ten. Who of you would like to move the dots? "Who can tell me the subtraction task?" "Find the task on the work sheet and fill in the result."	Solution of an exemplary, simple subtraction task
	Further subject		
material		"Today we will pay attention to our language. Not only solving the tasks but pay attention how to speak in the right way".	Stressing the importance of language

Sensitivity of language and	Connection of materials, figures and language	"Yes, five and five" (**shows two hands**).	Combining different forms of visualisation
		"I tell you, what to do: **I take away two dots from the ten**. Who of you would like to move the dots?"	Establishing helpful phrases
	Asking for explanations	"Tell me, how can you see that there are ten dots without counting?"	
	Grammar correction		

Sensitivity of behavior, attention and cooperation	"In the last lesson, we solved subtraction tasks. We gave special attention to our language. You do remember Murat. You do remember Lutz."	Directly addressing children in a positive way

Table 1. classroom situation starter plenary

Even though the textbook addresses two different interpretations (taking away from 10 and completing to 10), the teacher focuses on the idea of taking away from 10. This reduction goes hand in hand with a material- and language-sensitive specification of how the tasks are to be dealt with allowing all children to get access to the content (putting, describing and noting easy tasks like "10 - …"). In this sense, the children practise themselves in using the material and describing the operation. At the same time, the teacher opens the requirements for the children with need for assistance by using the 20-field holistically and not—as recommended in the handbook for promoting children with mathematical learning difficulties—reducing to the 10-field.

Due to the sensitivity of differences in competencies the result of the coding is that the teacher combines the solving of the subtraction task with the estimation of quantities. Doing so, the situation offers a learning opportunity for children on a more basic level to discover the structure of the twenty ten frames, according to the power of five or by remembering the structure. In addition, the plenum shows elements of sensitivity of language and materials. The actions (steps of solving) are consequently demonstrated and verbalised. As linguistic model the teacher implements a phrase which the children are asked to use in further learning situation. The teacher asks the children for explanations for the results and stresses the importance of arguing in mathematics education. The explanations enable children to understand the result and to challenge their reasons. Children which seem to have difficulties to concentrate in the lesson are addressed positively and personally.

DISCUSSION

The analysed inclusive practices do not distinguish themselves by great differences in organising the lesson or changing the materials, but in a sensible use of tasks and learning occasions in common situations. This - and the focus on simple subtraction tasks in the interpretation of taking away – seems to offer learning opportunities for each child in the class. The combination of complexity reduction while preserving the holistic structure, provides the basis for the children's collaboration in the following common learning situation. Following the processing of the tasks specified in the textbook, the approach during the partner work is designed as an open task, which eventually leads to autonomous inventions of own tasks and their documentation by the children.

References

Cobb, P. (1998). Analyzing the mathematical learning of the classroom community. The case of statistical data analysis. In O. Alwyn (Ed.), *Proceedings of the 22nd conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 33–48). Stellenbosch, South Africa: University of Stellenbosch.

Häsel-Weide, U., & Nührenbörger, M. (2017). *Das Zahlenbuch. Förderkommentar Lernen zum 1. Schuljahr.* Leipzig: Klett.

Mayring, P. (2015). Qualitative Content Analysis. Theoretical background and procedures. In A. Bikner-Ahsbahs, C. Knipping & N. Premeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 365–380). New York: Springer.

Nührenbörger, M., Rösken-Winter, B., Ip Fung, Ch., Schwarzkopf, R. & Wittmann, E.C., Akinwunmi, K., Lensing, F., Schacht, F. (2016). *Design Science and its importance in the German mathematics educational discussion.* Rotterdam: Springer.

Nührenbörger, M., Schwarzkopf, R., Wittmann, E. Ch. & Müller, G. N. (Eds.). (2017). *Das Zahlenbuch 1*. Leipzig: Klett.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tatrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM Mathematics Education, 44*(5), 641–651.

Scherer, P., Beswick, K., DeBlois, L., Healy, L., & Moser-Opitz, E. (2016). Assistance of students with mathematical learning difficulties: how can research support practice? *ZDM Mathematics Education, 48*(5), 633–649.

Wittmann, E. C. (2001). Developing mathematics education in a systemic process. *Educational Studies in Mathematics, 48*(1), 1–20.

THEORY OF GEOMETRICAL THINKING IN ELEMENTARY TEXTBOOK: CASE STUDY OF JAPAN

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Students begin to learn geometry based on their own experiences and concrete shapes. However, they eventually must depart from this foundation and develop geometric thinking. Textbooks should be helping students along this process. To this end, we examine whether Japanese elementary school (Grades 1–6) textbooks provide an appropriate foundation for developing students' geometric thinking. Through an original quantitative analysis based on the theory of praxeology, we determine that while first- through fifth-grade textbooks do provide a solid background for developing geometric thinking, fifth- through sixth-grade textbooks do not.

INTRODUCTION

When learning geometry, students start from their own experience and with concrete shapes and figures. However, their understanding must eventually go beyond such concrete ideas and reach the so-called "logical" level(s) of geometric understanding (cf. van Hiele, 1958). Over the long term, geometric curricula should assist this process. In this study, we examine whether such progression is found in Japanese elementary school curricula using mathematics textbooks. In Japan, textbooks can be seen as intended and/or implemented curriculum because they are authorized by the Ministry of Education and teachers usually use all pages of a textbook.

To this end, we attempt to determine whether Japanese elementary school (Grades 1–6) textbooks are appropriate for developing students' geometrical thinking as their understanding moves beyond their own experiences and concrete shapes; here, the term "geometrical thinking" refers to one's understanding of geometric objects and their learning method on van Hiele's scale (e.g., at level-0, objects are concrete things, but at level-1 they operate on them then class shapes; Fuys, Geddes, & Tischler, 1988). The novelty of our study is not just in the determination of this feature, but also will help to develop a quantitative method (as we mentioned) for describing these textbooks' characteristics.

THEORETICAL FRAMEWORK

Our theoretical framework is "praxeology," proposed by Chevallard (Bosch & Gascón, 2014), that, as a sub-theory of ATD (anthropological theory of didactics), can describe knowledge in any *institution*. In short, according to praxeology, an institution's knowledge has two parts: the practical (praxis) and theoretical (logos). Praxis can be broken down further into two variables: type of tasks, T, and technique, τ. T represents the tasks human beings face in their daily lives (Rasmussen, 2016), while τ describes the way of doing T. The logos aspect of knowledge, meanwhile, consists of both

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technology, θ, and theory, Θ. θ explains, justifies, and/or produces τ, while Θ elaborates on the meaning of discourse, encompassing the whole network of understandings and justifications used to account for technology itself and its relation to other technologies (Rasmussen, 2016). Thus, a single praxeology (p) can be represented by $p = [T / \tau / \theta / \Theta]$. For our purposes, it provides us with an understanding of the *ecology* of geometrical knowledge in the institutions using the textbooks.

METHODOLOGY

To determine how Japanese textbooks approach the development of geometric thinking, we quantitatively analyzed geometrical problems in Japanese elementary mathematics textbooks published by Keirinkan *(Wakuwaku-Sansu* first through sixth grade) using the praxeological framework described above. These books are some of the most widely used textbooks in Japanese elementary schools (34.3% in 2011; cf. Zizitsushin, 2010),[i] making them a good basis for our study. Table 1 shows the units included in our analysis.

Grades	Units studied and page numbers
1	Various Shapes (pp. 30–37), Making Shapes (pp. 96–101)
2	Triangles and Quadrilaterals (pp. 40–46; 2B), Shapes of Boxes (pp. 92–97; 2B)
3	Circles and Spheres (pp. 34–43; 3A), Triangles (pp. 2–13; 3B)
4	Perpendicular Lines, Parallel Lines, and Quadrilaterals (pp. 62–81; 4A), Area (pp. 2–17l; 4B), Cubes and Cuboids (pp. 88–101; 4B)
5	Volume (pp. 16–27), Congruent Shapes (pp. 70–85), Area (pp. 118–133), Circles and Regular Polygons (pp. 188–199), Prisms and Cylinders (pp. 200–207)
6	Symmetric Figures (pp. 8–25), Area of a Circle (pp. 66–77), Enlarging and Reducing Geometrical Figures (pp. 100–113), Volume of Solids (pp. 154–159), Approximate Shapes and their Sizes (pp. 160–163)

Table 1. Units included in our study (Shimizu et al., 2015)

Using praxeology allows us to interpret each question as praxeology in terms of its problem(s), solution(s), and related activities (as shown in Figure 2, they are sometimes implied). For example, Figure 1[ii] introduces the formula for finding the area of a right triangle (students by this point have already learned how to find the area of a square and rectangle). If one problem asks students to undergo more than one activity, we assume there is more than one praxeology (e.g., there are two in Figure 1). There are two possible ways to find the solution (Mirai's and Tsubasa's); thus, there are two praxeologies.

There are three types of questions in the textbooks: 1) introductory question (shown by the image of young leaves), 2) question for constructing new knowledge (shown by the boxed problem number in Figure 1), and 3) exercises question (shown by the circle around the problem number). For this study, we focus on the former two, since the third, exercises, is only intended to help students memorize knowledge.

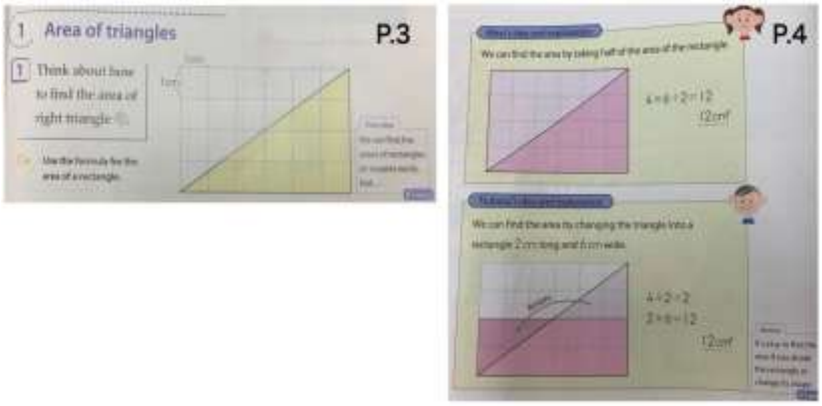

Figure 1. Problem of teaching students how to find a right triangle's area (Shimizu et al., 2012 (5B), pp. 3–4; the author integrated two pages and inserted labels)

In our analysis, we focused on logos, especially *Θ* which provides the basis of the logos aspect, called the praxeology. For example, when comparing the praxeologies of the two students, Mirai and Tsubasa, in Figure 1, each of whom solves the problem differently, we reach the following conclusions: both *T* is "to find the area of a right triangle", Mirai's τ was "making a rectangle and then cutting it" and using its formula for area and Tsubasa's τ is "cutting and rotating the triangle, to make a small rectangle" and then using its formula for area; both *θs* are equivalent transformation of any geometrical figure, etc. However, Tsubasa's *Θ* is empirical; his θ is justified and he understood that a right triangle is half a rectangle on the basis of its observed shape (i.e., visual evidence). On the other hand, Mirai's *Θ* is a geometrical theory (i.e., using the properties of rectangle) because her θ is justified and understood by the property of the rectangle, which is known to her. We refer to these two types of *Θ*s as I: Empirical and III: Property. For the purposes of our research question, Mirai's *Θ* is further along the progress on geometrical thinking, as it relies less on the student's own experiences with concrete shapes. Sometimes, θ and *Θ* cannot be clearly distinguished; however, in such cases, we use the same label because we want to know what is their praxeology's basis of logos. In Figure 2, one's τ is measuring tools (i.e., rulers), and θ and *Θ* are the tool's physical properties, social rules of units, etc. (in most case, we cannot distinguish this clearly). Thus, we named it II: Measurement. In Figure 3, *Θ* is found through arithmetical operations and/or algebraic theory; there are 12 unit squares but they should be calculated; thus, we named it IV: Operation. In Figure 4, there is no *Θ,* because all "facts" that are recognized by the students are appropriate; in that sense, this question might be used for fostering their sense (or intuitive knowledge) of the area of 1m^2. Thus, we labeled this type as X: Activities.

In this way, we categorized each textbook question based on which of the above five categories its Θ fit into. I, II, and X are considered at the "concrete" level while III and IV are at the "logical" level. In order to develop students' geometrical thinking in lower grades most questions should be in I, II, and X, while in higher grades, the proportion of questions in III and IV should increase.

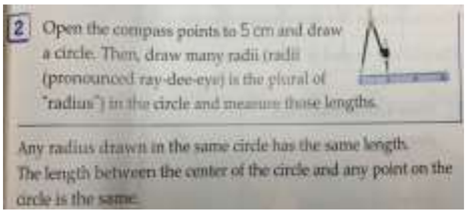

Figure 2. Measuring radii (Shimizu et al., 2012 (3A), p. 33)

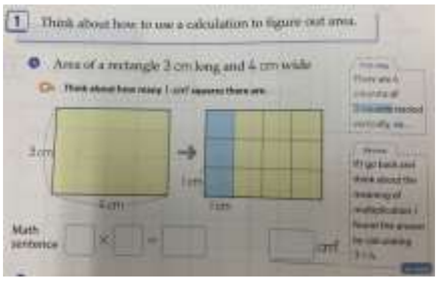

Figure 3. Finding the area of the rectangle (Shimizu et al., 2012 (4A), p. 86)

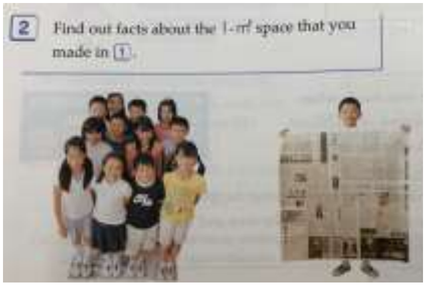

Figure 4. Understanding a 1 m^2 space (Shimizu et al., 2012 (4A), p. 90)

RESULTS

Table 2 shows the results of analysis. Because each grade's textbook has a different number of problems, Figure 5 shows the proportion of each question in each group.

Groups	Grades					
	1st	2nd	3rd	4th	5th	6th
I: Empirical	4	10	8	21	27	22
II: Measurement	0	9	6	15	12	18
III: Property	0	4	7	14	23	11
IV: Operation	0	0	0	7	12	4
X: Activities	7	5	4	6	10	2
Total	11	28	25	63	84	57

Table 2. Results of our analysis

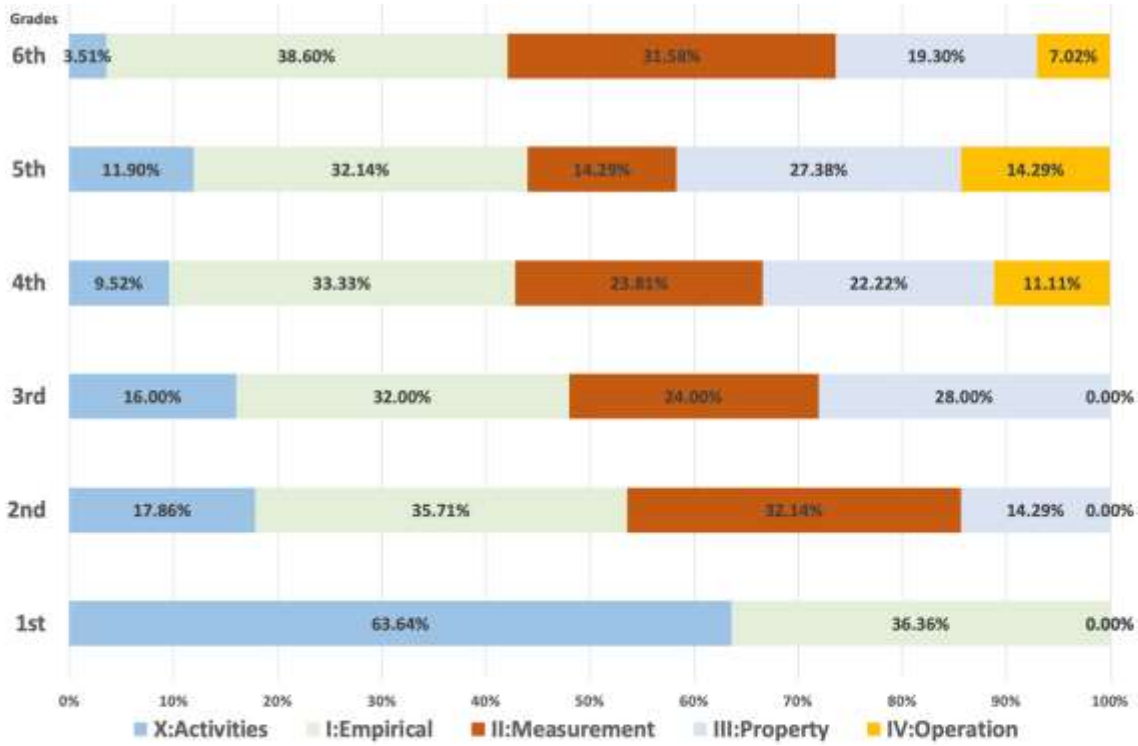

Figure 5. Proportion of questions in each group in each grade

Figure 5 shows that in Grades 1–5, overall textbooks are appropriate for developing students' geometrical thinking according, to van Hiele's scale: over time, questions with a concrete-level Θ (i.e., Groups I, II, and X) decrease and those with a logical-level Θ (i.e., Groups III and IV) increase. Because measurement with tools is learned in second grade, there are no Group II questions in the first grade, and no questions in Group IV are found until the fourth-grade textbook. However, the fifth- and sixth-grade textbooks did not meet our expectations. For example, questions with a concrete-level Θ increased between fifth and sixth grades and, in sixth grade, was almost equal to the proportion of questions concrete-level Θs in third grade (and higher than in first grade). This was due to two units: one on "symmetric figures" and the other on "enlarging and reducing geometrical figures" In the former, the number of questions in Groups X, I, II, III, and IV, respectively is 2, 14, 6, 2, and 0, and in the latter is 1, 4, 8, 1, and 0. Because these units are the first didactical opportunity for students to learn these types of knowledge, textbooks attempt to create opportunities for students to investigate and measure concrete figures. However, when considering the development of students' geometrical thinking and learning, this curricula must be reevaluated: either the teachers should adapt their methodologies to meet students' needs or textbook units should be rearranged.

CONCLUDING REMARKS

Our findings suggest that first- through fourth-grade Japanese textbooks are appropriate for fostering students' growing geometrical thinking (in the sense of our terminology), but fifth- and sixth-grade textbooks are inadequate. While our methodology is useful in revealing textbooks' characteristics, it does have some theoretical limitations. We cannot, for example, refer to the qualitative effects of these praxeologies. Our study is also limited in its exclusion of exercises, despite the fact that these occupy a certain proportion of textbooks. Future qualitative studies should attempt to overcome these limitations. In addition, we have no theoretical criterion for deciding Θ because praxeology merely provides a perspective for understanding knowledge in institutions, but as of yet there are no theoretical or objective criteria.

Future research should expand this work to additional textbooks. In addition, our methodology focuses specifically on geometrical learning in Japan; future work should extend our methodology to analyze other fields and other countries.

Acknowledgement

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References

Bosch M., Gascón J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds). *Networking of theories as a research practice in mathematics education*. Cham: Springer.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education, Monograph, 3,* 1–196.

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: Didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education, 19*(4), 301–324.

Shimizu, S. et al. (2012). *Fun with MATH for elementary school* (Grades 3A, 4A, 5B). Osaka, Keirinkan.

Shimizu, S. et al. (2015). *Wakuwaku Sansu* (Grades 1, 2A, 2B, 3A, 3B, 4A, 4B, 5, 6) (in Japanese). Osaka, Keirinkan.

van Hiele, P. M., van Hiele-Geldof, D. (1958). A method of initiation into geometry at secondary schools. In: H. Freudenthal (Ed.), *Report on methods of initiation into geometry*. Groningen: J. B. Wolter.

Zizitsushin. (2010). Status of adoption of elementary school textbook for 2011 – Report of Ministry of Education (in Japanese), *Naigaikyoiku*, Zizitsushinsha.

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[i] There are 6 authorized elementary mathematics textbooks in Japan; approximately 70% of schools have adopted Keirinkan and Tokyo-shoseki.

[ii] For readers' convenience, we refer here to English versions of the Keirinkan textbooks in Figures 1-4 (Shimizu et al., 2012). However, because the English version is older than the Japanese, while it is similar it varies in some particulars.

COMPETENCIES AND TEXTBOOK DEVELOPMENT: A THREE-DIMENSIONAL CONTENT MODEL ENACTED IN THE DANISH TEXTBOOK SERIES MATEMATRIX FOR GRADES K–9

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Curricula around the world make more and more use of goals trying to capture different kind of processes for the students to master. For mathematics education in Denmark, these ambitions have been described in terms of a set of mathematical competencies. However, bringing such competencies into the actual teaching practices has proved challenging. Matematrix is a Danish mathematics textbook system for grades k-9 designed to support the mathematics teachers in facing this challenge. In this paper, I – as one of the designers and authors of the textbooks – present one of the key elements in this endeavour: A three-dimensional content and objectives model combining mathematical competencies, mathematical core concepts and grade level. Following that, I exemplify the use of the model at three different levels of textbook design: The structuring of the content for the books in general, the focal points for each chapter in the various books and the development of tasks for a specific chapter.

INTRODUCTION

In the decade 1998–2008 I was one of the leading persons in the shaping and writing of a new series of mathematics textbooks, *Matematrix*, for the compulsory Danish "folkeskole", i.e. grades k–9. The first three years before the publication of the first book in 2001 were spent on deciding on and didactically designing the fundamental characteristics of the new books to come.

That process took place parallel to my involvement in the so-called KOM Project, where a set of mathematical competencies was proposed as a key element in the development of mathematics education in Denmark. Hence, one of the main ambitions for the new textbook system became an attempt to systematically facilitate the incorporation of mathematical competencies as a key element in mathematics education for grades k-9 in Denmark.

In this paper I will concentrate on describing how this ambition was fleshed out when developing a model for the content and objectives of the various parts of the textbooks. First, I will shortly introduce the KOM framework. Then I present the model itself, and finally I exemplify its use in the development of the actual textbooks.

THE KOM PROJECT, COMPETENCY AND MATHEMATICAL COMPETENCIES

The core of the KOM Project, running from 2000–2002, was to identify, explicitly formulate and exemplify a set of mathematical competencies as independent

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 197–202). Paderborn: Universitätsbibliothek Paderborn.

dimensions in the spanning of mathematical competence, cf. Figure 1. Niss & Højgaard (accepted) gives an updated account of the KOM framework, whereas Niss & Højgaard (to appear) provides a more thorough presentation and analysis of the project and an English translation of the original report.

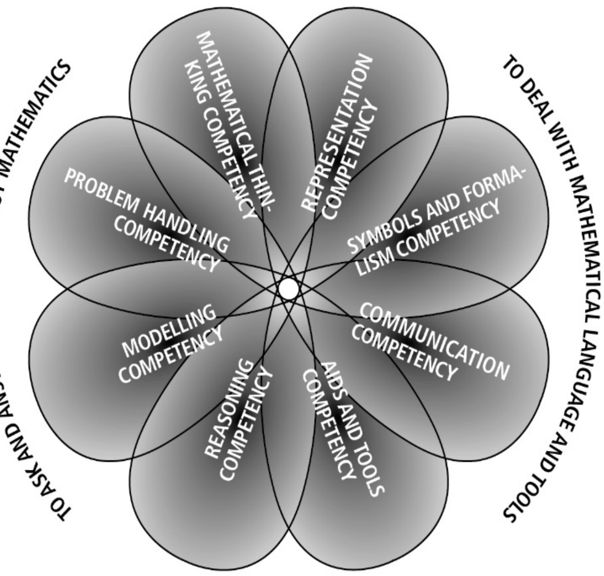

Figure 1. A visual representation—the "KOM flower"—of the eight mathematical competencies presented and exemplified in the KOM report (Niss & Højgaard, accepted).

Such a set of mathematical competencies has the potential of replacing the syllabus as the focus of attention when working with the development of mathematics education, simply because it offers a vocabulary for a focused discussion of what it means to master mathematics (Blomhøj & Jensen, 2007; Jensen, 2007).

The definition of the term "competence" in the KOM report (Niss & Jensen, 2002, p. 43) was semantically identical to the one I use: Competence is someone's insightful readiness to act in response to the challenges of a given situation (cf. Blomhøj & Jensen, 2003). In definite form, a mathematical competency is consequently defined as someone's insightful readiness to act in response to a certain kind of mathematical challenge of a given situation.

A MODEL OF TEXTBOOK CONTENT AND LEARNING OBJECTIVES

Following the approach of the KOM Project, a pivotal part of the endeavour to systematically facilitate the incorporation of mathematical competencies as a key element in mathematics education is to separate mathematical competencies and subject matter areas as two independent dimensions of content (Højgaard, 2012; Niss & Højgaard, to appear). Subsequent research and development work supported the importance of such an approach (Højgaard & Sølberg, 2019).

Hence, it was decided that a set of mathematical competencies—slightly different from the ones in the KOM report due to research on putting the competencies into educational practice (Jensen, 2007, pp. 264–265)—on the one hand and a core syllabus consisting of a set of 27 fundamental mathematical concepts derived from the curriculum on the other hand should act as two independent dimensions in the spanning of the content and underlying learning objectives for all the *Matematrix* books for grade k–9. Analytically this can be used to create the three-dimensional model depicted in Figure 2.

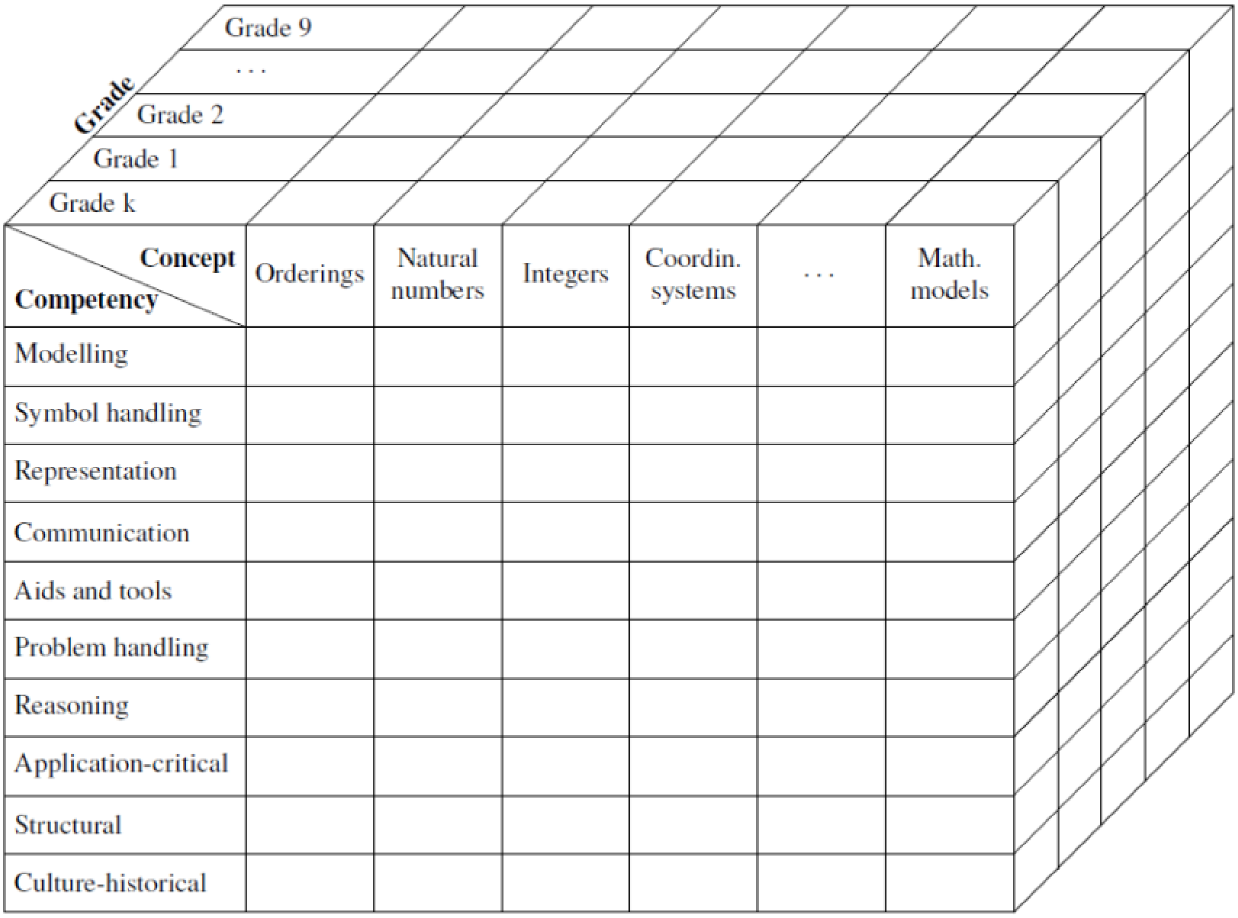

Figure 2. A three-dimensional competencies × concepts × grade model for deciding on the content and objectives of the various parts of the books in the textbook series *Matematrix*.

EXAMPLES OF PUTTING THE MODEL TO WORK

To make it a functional didactical tool, the three-dimensional model was used to generate two two-dimensional matrix models. The first step was to “lift off the ceiling” of Figure 2 to get a large matrix structure combining the 27 fundamental concepts with the 10 grades (Jensen, 2001). This model was used to structure discussions of the emphasis on and progression in the intended work with each concept, specified by deciding on the headings and proposed conceptual learning objectives of each chapter in each book. As an example, the concept of function is introduced in a chapter in the book for grade 7. Prior to that, the chapter *Relationships* in the book for grade 6 (Gregersen et al., 2008a) focuses on relationships between variables more generally and everyday like, and following that a chapter in the book for grade 8 introduces linear functions.

These conceptual choices for each book were then combined with the set of competencies to form a second so-called competency matrix for each book; competencies × chapter headings. As an example, Figure 3 shows the competency matrix for the textbook for grade 6, given in the accompanying teachers manual (Gregersen et al., 2008a,b).

In the developmental process, these competency matrices have been used as a vehicle to maintain a strong focus on the mathematical competencies. This has been fleshed out on both a chapter and a task design level. Some of the chapters have primarily been decided on and developed with a specific competency in mind. As an example, the chapter *Reality and mathematics* in the book for grade 6 is devoted to mathematical modelling, and he explanations in the center of the chapter (Gregersen et al., 2008a, pp. 130–131) is about the mathematical modelling process, not specific mathematical concepts.

For task design purposes, the competency matrices have been used to decide on and communicate which 2-3 competencies that were explicitly focused on and proposed as learning objectives for each chapter in the book, cf. Figure 3. To make these decisions more concrete and binding, the guidelines for each chapter in the teacher's manual are initiated by the list of proposed learning objectives (e.g., Gregersen et al., 2008b, p. 30):

- 2–4 with a conceptual focus, e.g., "[…] develop an understanding of, what it means that there is a relationship between different incidents and magnitudes" (Gregersen et al., 2008b, p. 30, author's translation), stemming from the concepts × grades model, and
- 2–3 competency objectives, e.g., "[…] represent mathematical relationships in different ways and gain experiences with their different strengths and weaknesses" (Gregersen et al., 2008b, p. 30, author's translation), stemming from the competencies × conceptual chapter headings model.

As a design principle strengthened in the ongoing revision of the entire *Matematrix* book series, each of the stated competency objectives are accompanied by a list of tasks (from the chapter in focus) explicitly designed with that objective in mind, and this

Chapter / Competency	Algebra	Movements	Equations	Drawing	Fractions	Percentages	Relationships	Statistics and probability	Formulas	Reality and mathematics
Modelling		Patterns as geometric models. Using movements as a tool for constructing patterns.		Drawings as geometric models, incl. a choice between three drawing models.			Tables, graphs, equations etc. as models of relationships from reality.		Construction of formulas as models of various relationships.	Aritmetic tasks, formulas and drawings as modelling tools.
Symbol handling	Letters as place holders for unknown numbers. Preparing for the concept variable.		Decoding, construction and transformation of equations.		A fraction understood as a relation between numbers and as a repr. of a number in itself.	Decoding of and calculations with percentages, fractions and decimal numbers.	Linguistic description of various symbolically given relationships.		Decoding existing formulas. Construction of one's own formulas.	Construction and use of one's own arithmetic tasks and formulas.
Representation	Unknown numbers represented by numbers.		Unknown numbers represented by letters in equations.	Various drawings as different representations of the same object.	Transformation from e.g. fraction to decimal number.	Percentages as representations of scales.	Mathematical relationships represented by e.g. everyday language, equations and tables.		Formulas as a way to represent relationships.	Choosing between arithmetic tasks, formulas and drawings as modes of representation.
Communication		Comm. about the elements in various patterns, and about how geo. fig. can be moved.				The communicative power of percentages, e.g. when comparing fractions.	Comm. about various relationships between unknown quantities.	Comm. about the use of statistics and probability.	Comm. about the relationships between unknown quantities given by various formulas.	Comm. about models made by one self and others and the interpretation of their results.
Aids and tools		Using pairs of compasses, coordinate systems, protractors, mirrors, etc.		Using isometric paper and exact tools to draw long, straight lines.	The calculator as a tool for transforming fractions to decimal numbers.	The calculator as a tool for transforming scales to percentages.			Decoding and constructing formulas in spreadsheets.	
Problem handling			*Solve the equation* $x+1=x$.		*Can you calculate with all fractions?*	*What is the smallest number you can get when finding percentages of something?*		*What is the mean of five subsequent integers?*		
Reasoning	Reasoning about the validity of arithm. rules, e.g. opposite numbers and arithm. operations.	Reasoning about patterns and symmetri as geometric properties.	Relating to statements about equations and their solution (testing possible solutions).			Deciding if statements about calculations of percentages are true or false.		Reflections on the quality of various arguments based on statistics and probability.	*Are these formulas the same?* $E=2\cdot L$ *and* $L+L=E$	
Application-critical				Critically considering the choices a designer/painter necessarily has to make.			Experiments as a means to examining hypotheses about relationships.	Assessing the fairness of various arguments based on statistics and probability.	Critically relating to other peoples construction of formulas.	Critically relating to other peoples models and interpretation of results.
Structural	On opposite numbers and opposite arithmetic operations.	Symmetri as a geometric property.			Different fractions as representations of the same rational number.					
Culture-historical			The history of the equation.				Time zones as a way to define time different places on the Earth.			Currencies and rates of exchange in other countries.

Figure 3. An example from grade 6 of the competency matrix – competency objectives × chapter headings – accompanying each book in the mathematics textbook series *Matematrix* (Gregersen et al., 2008b, p. 13, author's translation).

competency potential is addressed in the comments to each task. As an example, the competency objective given above regarding strengths and weaknesses of different representational modes has been used to generate the following task (Gregersen et al., 2008a, p. 134, my translation): "Use different mathematical tools to work with question a-e. […] c. How can one draw a sunset? […]"

References

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and its Applications*, *22*(3), 123–139.

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? Experiences with using a competence perspective on mathematics education to develop the teaching of mathematical modelling. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Applications and modelling in mathematics education: The 14th IMCI Study* (p. 45–56). New York, USA: Springer.

Gregersen, P., Jensen, T. H., Petersen, L. K., & Thorbjørnsen, H. (2008a). *Matematrix 6*. Copenhagen, Denmark: Alinea.

Gregersen, P., Jensen, T. H., Petersen, L. K., & Thorbjørnsen, H. (2008b). *Matematrix 6 – Lærervejledning* [Matematrix 6 – Teachers manual]. Copenhagen, Denmark: Alinea.

Højgaard, T. (2012). Competencies and the fighting of syllabusitis. In ICME 12 LOC (Ed.), *Pre-proceedings of ICME 12* (p. 6412–6420). Seoul, South Korea: ICME 12 LOC.

Højgaard, T., & Sølberg, J. (2019). Competencies and curricula: Two case stories of two-dimensional curriculum development. *European Journal of Science and Mathematics Education*, *7*(1), 50–60.

Jensen, T. H. (2001). *Kerneområdematrix, version 1* [Core concepts matrix, version 1]. Unpublished working paper from the development of the textbook series *Matematrix*.

Jensen, T. H. (2007). Udvikling af matematisk modelleringskompetence som matematikundervisningens omdrejningspunkt – hvorfor ikke? [Developing mathematical modelling competency as the hub of mathematics education – why not?]. Roskilde, Denmark: IMFUFA, Roskilde University. Ph.D. dissertation. Available from the authors homepage.

Niss, M., & Højgaard, T. (accepted). Mathematical competencies revisited. *Educational Studies in Mathematics*.

Niss, M., & Højgaard, T. (to appear). *Mathematical competencies in mathematics education: Past, present and future*. New York, US: Springer.

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring: Idéer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching in Denmark]. Copenhagen, Denmark: Ministry of Education.

REPORT ON THE DESIGN AND REALISATIONS OF THE WORKSHOP 'WRITING MATHS TEXTBOOKS'

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Report on the main ideas, the design, the practical and theoretical components and the realisations of the workshop 'Writing Maths Textbooks' which the author developed and run in 2016 and 2018 on behalf of the GIZ (Deutsche Gesellschaft für Internationale Zusammenarbeit) to teach future textbook writers how to write Maths Textbooks. Because of acts of war the workshop was delivered in 2016 per skype. A second face to face realisation of the workshop took place 2018 at the Hanoi National University of Education (HNUE).

INTRODUCTION

Genesis and Purpose of the Project

In 2016 the German government organisation GIZ (Deutsche Gesellschaft für Internationale Zusammenarbeit) commissioned me to run a workshop for future textbook authors of the Republic of Yemen on compiling maths textbooks. Because of acts of war the workshop was delivered in 2016 per skype. A second face to face realisation of the workshop took place 2018 at the Hanoi National University of Education (HNUE). I could not find any reference for such a project. Therefore I designed it myself on the base of my experiences as author and editor of about 25 maths textbooks in just as many years. Of course, you cannot teach novices how to write good books in a week. But you can start to give them feeling and knowledge about the dimensions of this activity and to let them practise their first steps to become authors.

Importance of Textbooks

Beside all materials and resources, the textbook is the most influential medium, the main channel and the main resource for students and for teachers. Textbooks embody and incorporate school mathematics both literally and figuratively speaking. Textbooks generate and grant

- Steadiness
- Consistency
- Perpetualness
- Continuity
- Sustainability

for students and for teachers. There is no Teaching 4.0 or Learning 4.0 or higher.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 203–208). Paderborn: Universitätsbibliothek Paderborn.

One crucial point to assist and to support learning is to offer suitable activities, situations and materials to get mathematical experiences and to reflect them. And the teachers often do not have the time and the resources to supply all of that. Experiences (for example in Kosovo) show me that teaching without rich textbooks does not work properly. Textbooks can/may induce and/or offer various styles of teaching and methods of learning. On one hand textbooks should not only support one style of teaching (because teachers are different). But on the other hand, textbooks should support an open style of teaching. The task of textbook writing is not just to wrap some content nicely and to hand it over to students rather than to create an environment for the students

- to make experiences, to discover and to construct their own knowledge and ability
- to support active learning and productive practising
- and to state results in an easy understandable way and a summary for revising, rereading, relearning and remembering

assisted and supported by their teachers.

The life span of a maths textbook is quite short, certainly not much longer than ten years. The environment, the didactics, the visual demands and even the students, teachers and parents are changing in such a period to an extend that it is necessary to rewrite the textbooks or to write new ones

The Art of Writing Textbooks

Even so you need talent and sprachgefuehl to write textbooks there is a lot to learn. Only learning by doing is here a waste of time. Practising it in a group under guidance is useful and effective. It is very difficult to write a text which looks as it would have been quite easy and natural to write it just like that.

THE DESIGN AND FRAMEWORK OF THE WORKSHOP

Theoretical Components

There have been six lectures based on PowerPoint presentations in English and the first language (for example Arabic or Vietnamese) as well.

- Writing Maths Textbooks - A first Approach
- Importance and Consequences of the current Curriculum Research and Practice for Textbook Writing
- Textbooks: General Decisions
- Features of Textbooks
- Elements of Textbooks
- Lesson and Video Studies

and two smaller additional ones as a kind of resume:

- Essential Criteria to Evaluate Textbooks
- From Curricula to Textbooks

Example of a lecture

The lecture 2 on the "Importance and Consequences of the current Curriculum Research and Practice for Textbook Writing" is dealing with the concepts of 'competencies' and 'leading ideas'.

- The meaning of competencies and their phrasing is discussed thoroughly looking in curricula of different national programmes and their Danish origin by Mogens Niss. Furthermore, the relations between the competencies and so called 'operators' in textbook tasks are debated using examples from different textbooks (for instance the 'Zahlenbuch' by Erich Ch. Wittmann and others).
- The concept of leading ideas/big ideas/strands/domains in various arrangements are reconsidered in connection with the ideas of a spiral curriculum by Jerome Bruner for primary and secondary maths education as well.
- Finally, the seven goals of general and mathematical education according to Hans Werner Heymann are introduced as wider and broader and more substantial perspectives in this context: Preparation for Later Life / Promoting Cultural Competence / Development of an Understanding of the World / Development of Critical Thinking / Developing a Willingness to Assume Responsibility / Practice in Communication and Cooperation / Enhancing Students' Self-Esteem.

Other lectures broach teaching and content related issues such as the balance of instruction and construction or the nature of mathematics (applications and modelling/ contextualisation and decontextualisation).

Practical Components

I designed and prepared

- Missions (set ahead of the workshop)
- Assignments
- Theory tasks (on the basis of the Reader; see below)
- Maths problems
- Word problems and
- A Final assessment

for the practical work.

Examples of assignments

To learn and practise writing the participants should pen different kinds of texts for example letters and rules of a game.

Answer one of these two letters twice carefully

1) by explaining a solution to her problem

2) by giving her hints how to tackle the problem.

Dear Miss Trainee,

I cannot go to sleep tonight because I am so angry, and my parents do not want to talk to me. I was today invited to Berta's birthday party. Hanna and Klaus and Jakob and Christine have been there as well. Actual it was quite nice. But in the end there was a lottery. Every child was allowed to draw a lot out of a hat, and I was the last. This is so unfair. Of course, I got only the consolation gift. With my birthday party I want to get the first lot. Didn't you say one day it is always fair in mathematics? But not on the birthday party of Berta.

Love Anna

Dear Miss Trainee,

You are so wonderful in explaining. That is so reassuring. Yesterday I played with Bernd. He got a dice. So, we did a game: We threw the dice three times. When there was at least one six I got a candy. Otherwise he got one. I think he got more candies than me. But one must be able to lose.

When will you write me again?

Love Anna

Give a list of outdoor games for young children. Write down the instructions for an outdoor game in an easy understandable way.

The Reader

I compiled a selection of excerpts of important articles and essays to get a common background for the participants of the workshop. It is not meant to determine special theories as base of all textbook writing rather than to enable the participants to look and to discuss texts from different perspectives. From the table of content:

<ins>Learning Theories:</ins> General (Learning Theories, On Teaching for Understanding, Discovery Learning); Piaget's Work; Bruner (The Process of Education, Bruner's Theory, Stages, Spiral Curriculum, Modes of Representation)

Open Problems: Activities, Open Problems, Example Project

Best Practice (for primary schools): Zahlenbuch (Wittmann/Müller); Primary Maths Courseguide (Oxford); Ruf-Gallin (Dialogic Learning, I – You – We –Principle)

Global Testing: TIMSS, PISA, On Testing Culture

THE ORGANISATION AND THE PRAXIS OF THE WORKSHOP

The workshop is designed for 5 days. In case of a final assessment to choose the most talented authors an additional day is needed. Because of the practical work, the marking and the corrections the workshop is restricted to not more than 20 active (!) participants ('trainees') in 4 or 5 groups. The lectures are open for a wider audience. With the exception of the first day the daily schedule looked like that:

09:00– 9:30	Revision and Resumé of the last day by Group x and Feedback from the consultant on the last day
09:30–10:30	Lecture
10:30– 11:00	Break
11:00–12:00	Reading and Discussing one of the excerpts from the Reader prepared and moderated by Group x
12:00–13:30	Lunch break
13:30–15:00	Practical session
15:00–15:30	Discussion of trainees' work
15:30–16:00	Presentation of the missions' results/discussion of the day with implications on next one
16:00–17:30	Home review of the trainees' work by the consultant

The female and male participants have been chosen by the local organizers: teachers, PhD-students, lecturers, educators from all over the country. In the end they got a certificate for attending the workshop successfully.

Some but important

RESULTS AND EXPERIENCES - FURTHER STUDIES AND PROJECTS

From the trainees

- Five days is a short period.
- You can not talk to your readers.
- You have not got an index finger to show them your pictures or tables.

From the lecturer:

- Five days is a short period.
- Maths is highly compressed, and it is one of the main and difficult tasks of an author to decompress maths. It takes time to slow down and work and write thoroughly.

Having developed and run the workshop with satisfying feedback I will improve and offer it to other nations and countries, universities and other educational institutions.

NOT ADRESSED PROBLEMS AND UNANSWERED QUESTIONS

I am aware that essential questions – for instance

- Which considerations should be made when exporting workshops in writing maths textbooks to less developed countries?
- What is the theoretical background or framework of such (postcolonial?) projects?

cannot be tackled on these 6 pages.

A COMPARISON OF PAGE SAMPLING METHODS FOR MATHEMATICS TEXTBOOK CONTENT ANALYSIS

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We utilize four sampling methods (simple random sampling, systematic sampling, stratified random sampling, and cluster sampling) to select pages from ten textbooks, resulting in forty samples of pages. For each sample, we estimate the percentage of pages containing statistics and the number of statistics tasks within each textbook. These estimates are compared with the actual values obtained by examining every page of each textbook. Within these data, stratified sampling produces the least error in the number of statistics tasks, and all confidence intervals for the percentage of pages containing statistics based on stratified sampling contain the actual percentage. For these variables, a census is preferable to other sampling methods.

INTRODUCTION

A variety of sampling methods have been used in mathematics textbook research. Examples include taking a census of pages by examining every page in the textbook (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), examining a sample of pages, and examining all of the pages in a sample of lessons (Bieda, Ji, Drwencke, & Picard, 2014) or chapters. These sampling methods may incorporate random selection, systematic selection, or purposeful selection.

Each method has costs and benefits. For example, Jones et al. (2015) conducted a census of the pages of five textbook series with five books in each series. A total of 17,688 pages were examined to determine the distribution of statistics tasks within the textbooks. This method of examining every page required a large investment of time, but the results came with a high level of confidence. The researchers could be very certain that the percentage of pages containing statistics and the number of statistics tasks within each textbook were accurate.

The purpose of this study is to determine whether the cost of time in a census is justified by the benefits of confidence in the results. One may wonder whether examining a smaller sample of pages or chapters would provide similar results without sacrificing confidence in the results.

Sampling Methods

In this study, we utilized four different methods for sampling pages, and compared the results to those of the census in the study by Jones et al. (2015). The four methods, or types of samples, are simple random sample, systematic sample, stratified random sample, and cluster sample. Below, we define the four methods, and explain how they are applied to sampling textbook pages.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 209–214). Paderborn: Universitätsbibliothek Paderborn.

In a *simple random sample*, each possible sample of a given size has an equal likelihood of being selected. For our study, the sampling unit is pages contained in chapters of the textbook. The pages are selected without replacement, so that each page is included in the sample at most once.

A *systematic sample* uses an algorithmic method to select pages for the sample. In our study, textbook pages are numbered. The sample is formed by selecting a page at random from the first m pages, and then including additional pages so that the difference between subsequent page numbers is equal to m.

In a *stratified random sample*, the population is partitioned into strata. For our study, the strata are chapters. A simple random sample of pages is selected from each chapter, where the number of pages in the sample from a particular chapter is proportional to the number of pages in that chapter. The pages selected from each chapter form the sample.

For the *cluster sample*, the population is partitioned into clusters. In our study, the clusters are chapters. A simple random sample of chapters is selected, and all the pages from those chapters are included in the sample.

Research Question

In this paper, we address the following research question: How closely does the use of each of four sampling methods (simple random sampling, systematic sampling, stratified random sampling, and cluster sampling) match the true values obtained from a census, with respect to the percentage of pages containing statistics and number of statistics tasks in the textbook? By examining this question, we hope to provide insight into whether any of these four methods may perform better than others, and potentially as well as a census, with respect to the variables of percentage of pages containing statistics and the number of statistics tasks in the textbook.

METHODOLOGY

Textbook Selection

We purposefully selected two of the five textbook series that Jones et al. (2015) examined. One series, *Math Connects* (Altieri et al., 2009) was developed by a commercial publisher, and the other, *Math Trailblazers* (Wagreich et al., 2008) was developed with funding from the National Science Foundation to address mathematics education reforms advocated by the National Council of Teachers of Mathematics (2000). We include the five textbooks from each series that were written for grades 1 to 5 (i.e., students ages 6 to 11 years). Both series structured the content into chapters, with a particular mathematical content focus for each chapter. Therefore, this study examines ten different textbooks.

Sampling Methods and Coding

For each of the ten textbooks, we utilize four sampling methods (simple random sampling, systematic sampling, stratified random sampling, and cluster sampling) to select pages, resulting in forty samples of pages. Yamane (1967) provided a formula to determine the "sample size for specified confidence limits and precision when sampling attributes in percent" (p. 886). In our study, the level of precision is ±5%. For each textbook, the total number of pages N is used to determine the sample size n (for all but the cluster sample) using the formula $n = N/(1+N(0.05)^2)$.

For the simple random sample (Sim), we use the statistical software environment R (R Core Team, 2018) to randomly select n of the N pages. The systematic sample (Sys) comprises every page with a page number closest to $j + k(N/n)$, where j is a randomly selected integer between 1 and N/n, and k is an integer from 0 to n–1, inclusive. The stratified random sample (Str) is formed by using N_t, the total number of pages in chapter t, to determining the stratum sample size $n_t = N_t(n/N)$. We use R to select a simple random sample of n_t pages from chapter t, where $n_t = N_t(n/N)$. The pages in these simple random samples combine to form the stratified random sample. For the cluster sample (Clu), we use the k-means clustering package within R to select a simple random sample of chapters. All the pages in those selected chapters form the cluster sample.

Jones et al. (2015) examined a number of variables related to the statistical content of textbooks. In this study, we focus on two of those variables: including the percentage of textbook pages containing statistics, and the number of statistics tasks in the textbook. Therefore, for each page in each sample, we examine the census data set and determine whether or not the sampled page contains statistics, and the number of statistics tasks on the page.

Calculating and Evaluating Estimates

For each sample, we determine the percentage of pages in the sample that contained statistics. We then used R to construct a 95% bootstrap confidence interval (CI) for this percentage, based on 10,000 bootstrap samples. The bootstrap process uses resampling with replacement and provides one measure for how well the estimated percentage of pages from a given sample matches the actual percentage. In general, narrower CIs are preferable. We also note whether the CI contains the actual percentage of pages containing statistics, as reported in Jones et al. (2015), as well as the error in the estimate (the difference between the estimate and the actual percentage of pages containing statistics).

With respect to the number of statistics tasks, we estimate the total number of statistics tasks in the textbook by multiplying the total number of pages by the ratio of the number of statistics tasks in the sample to the number of pages in the textbook. We then compare the estimate to the actual number of statistics tasks to determine the error in the estimate for the number of statistics tasks in the textbook.

RESULTS

We obtain four samples from each of the ten textbooks. Table 1 provides the data related to the four samples collected from a single textbook, *Math Connects* Grade 1.

	Sampling Method				
	Sim	Sys	Str	Clu	Census
Sample size *n*	239	239	240	310	*N* = 593
Number (percentage) of pages containing statistics in sample	12 (5.0%)	15 (6.3%)	12 (5.0%)	36 (11.6%)	40 (6.8%)
95% bootstrap CI for percentage of pages containing statistics	2.9% to 7.5%	3.8% to 8.8%	2.9% to 7.5%	8.8% to 14.8%	
Number of statistics tasks in sample	30	48	39	113	123
Estimated number of tasks in textbook	74	119	96	216	

Table 1. Results based on samples of pages from *Math Connects* Grade 1 textbook

This textbook has a total of 593 pages across 16 chapters, which means the sample should include 239 pages (Yamane, 1967). Note that the percentage of pages in each sample is within 5% of the true percentage of 6.8%. The 95% bootstrap CIs all have widths of 6% or less, with the simple random and stratified random samples having the narrowest CIs. The percentage of pages containing statistics estimated from the systematic sample is the closest to the actual percentage; the estimated number of tasks for the textbook is also closest to the actual for the systematic sample. For this particular textbook, the cluster sample overestimates both the percentage of pages containing statistics and the number of statistics tasks in the textbook. This is most likely due to selecting the chapter on statistics content as one of the clusters. Consequently, the cluster sample for this textbook contains relatively more statistics content than other samples that do not select every page of every chapter.

Our study examined a total of 40 samples, with ten samples of each type: simple random, systematic, stratified random, and cluster. We now examine the data in terms of sampling method. In Figure 1, we summarize the widths of the 95% bootstrap CIs. Each boxplot is based on the widths of the ten bootstrap CIs for a particular sampling method. The cluster samples have the lowest median width of about 7%, but the widths of the CIs vary from approximately 5% to 10%. None of the four methods appears to stand out as markedly different from the others with respect to the width of the 95% bootstrap CI.

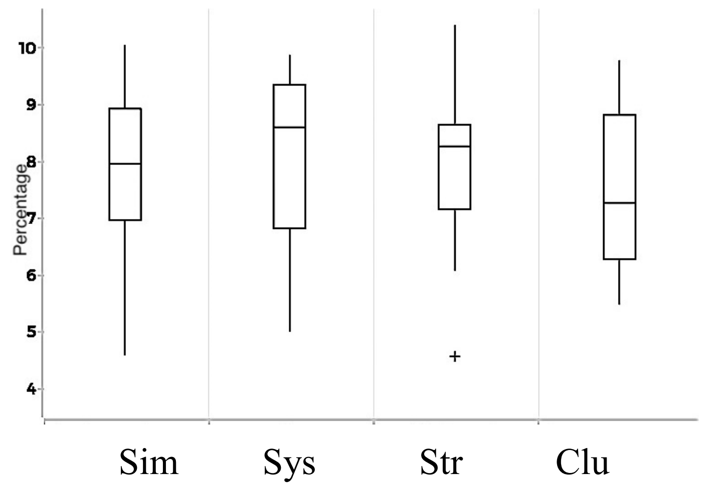

Figure 1. Widths of confidence intervals for percentage of textbook pages containing statistics, by sampling method.

These four sampling methods are also compared in terms of the error in the estimates of the percentage of textbook pages. Figure 2(a) contains boxplots representing the distribution of errors for percentage of pages containing statistics for each sampling method. The systematic and cluster samples both have medians near zero; the cluster samples have the most variability. The simple random and stratified random samples appear to underestimate the percentage, as the errors tend to be negative.

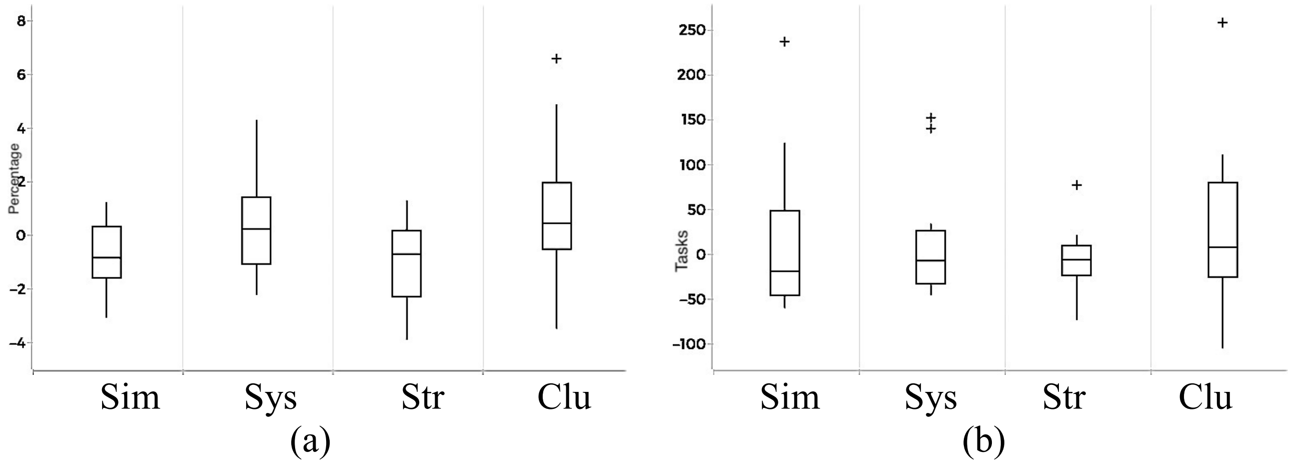

Figure 2. Errors in estimates of (a) percentage of textbook pages containing statistics and (b) number of statistics tasks in the textbook, by sampling method

Regarding the errors in estimates of the number of statistics tasks in the textbook, Figure 2(b) shows that each method has produced an error greater than 50 tasks, and some as large as 250 tasks. The actual number of statistics tasks in these ten books varied from 113 to 539. The systematic and stratified random samples appear to have the least variability, although outliers exist for each method. It appears that this estimate is very sensitive to sampling conditions.

DISCUSSION AND IMPLICATIONS

Within these data, stratified sampling produces the least error in the number of statistics tasks, and all confidence intervals for the percentage of pages containing statistics based on stratified sampling contain the actual percentage. Therefore, it would be reasonable to use a stratified random sample when estimating the percentage of pages with some attribute. At the same time, this sampling method overestimated the total number of statistics tasks by more than 100 in two textbooks.

While each sampling method may have costs and benefits, it is important to understand the research question and variables to be investigated, and utilize a sampling method that will best address the question. In the case of Valverde et al. (2002), a census was most appropriate to provide for a rich description and comparison of textbooks. In the Bieda et al. (2014) study, the decision to use a systematic sample "greatly enhanced the feasibility of the study," (p. 79) as there were over 25,000 problems to potentially analyse.

Based on these results, it is difficult to state that any of these four sampling methods would consistently provide estimates close to what Jones et al. (2015) found using a census. This is likely due to the lack of homogeneity in the distribution of statistics tasks within textbooks, as textbooks are typically organized into content-based chapters. Future studies could compare sampling methods for textbook features that are more likely to be homogeneously distributed, such as types of figures, discussion prompts, worked examples, or opportunities for reasoning-and-proving.

References

Altieri, M. B., Balka, D. S., Day, R. D., Gonsalves, P. D., Grace, E. C., Krulik, S., et al. (2009). *Math Connects*. Columbus, OH: McGraw-Hill.

Bieda, K. N., Ji, X., Drwencke, J., & Picard, A. (2014). Reasoning-and-proving opportunities in elementary mathematics textbooks. *International Journal of Educational Research*, *64*, 71–80.

Jones, D. L., Brown, M., Dunkle, A., Hixon, L., Yoder, N., & Silbernick, Z. (2015). The statistical content of elementary school mathematics textbooks. *Journal of Statistics Education, 23*(3).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

R Core Team (2018). *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna, Austria. Retrieved from http://www.R-project.org/.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book - Using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer.

Wagreich, P., Bieler, J. L., Goldberg, H., Kelso, C. R., Beissinger, J. S., Cirulis, A., et al. (2008). *Math Trailblazers*. Dubuque, IA: Kendall Hunt Publishing.

Yamane, (1967). *Statistics: An introductory analysis* (2nd ed.). New York: Harper & Row.

PRESERVICE TEACHERS' LEARNING TO USE EXISTING RESOURCES PRODUCTIVELY

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The capacity needed for using existing resources was explored and its potential components were identified previously. Building on the previous work, this study further explored the capacity needed for productive resource use in a mathematics methods course. The results indicate that all of the components identified were challenging to develop, and yet the most complex component is "steering instruction toward the mathematical points of the lesson." This suggests re-conceptualization of the capacity for productive use of existing resource use.

BACKGROUOND OF THE STUDY

Teacher capacity needed for productive resource use was explored based on a range of analyses of elementary teachers using various curriculum programs in the United States (Kim, in press). This exploration led to identify some important components of the capacity. In a mathematics methods course, these components were utilized to support preservice teachers (PSTs) to develop the capacity. This paper reports a preliminary investigation of the usability of this construct in helping PSTs use curriculum resources productively. The research question is: *How does incorporating the construct of teacher capacity for productive resource use in a mathematics methods course support PSTs to develop the capacity for productive resource use?*

THEORETICAL FOUNDATIONS

This study is based on Brown's (2009) notion of pedagogical design capacity (PDC), which defined as "a teacher's ability to perceive and mobilize existing curricular resources" (p. 29) in order to design instruction. Researchers have explored PDC by attending to teacher decisions on resource use (e.g., Leshota & Adler, 2018). Productive resource use needs a range of knowledge and skills, such as knowledge of content and curriculum (Ball, Thames, & Phelps, 2008). In my exploration of the capacity needed for productive resource use, I attended to specific tasks that teachers need to do. Consolidating a set of analyses on teachers' use of existing curricular resources, I identified five components/tasks that are important in productive use of existing resources: (1) identifying the mathematical points of individual lessons (within and across), (2) steering instruction toward the mathematical points, (3) recognizing affordances and constraints of the recourse they use, (4) using the affordances of the resources, and (5) filling in the gaps and holes in the resources (Kim, in press). These tasks are interrelated; one task is influencing and influenced by other tasks. For example, teachers can recognize affordances and constraints of resources when they clearly identify the mathematical point of the lesson.

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Productive resource use involves making and carrying out a rigorous instructional plan based on the resources in use in order to reach the instructional goal, i.e., students' learning of the target mathematics. Sleep's (2012) conceptualization of the work of teaching to the mathematical point helps examine how well teachers orchestrate instruction to support students to learn the target mathematics. She identified seven tasks for steering instruction toward the mathematical point: (1) attending to and managing multiple purposes, (2) spending instructional time on mathematical work, (3) spending instructional time on the intended mathematics, (4) making sure students are doing the mathematical work, (5) developing and maintaining a mathematical storyline, (6) opening up and emphasizing key mathematical ideas, and (7) keeping a focus on meaning. For each of these seven tasks, Sleep identified strategies and problematic issues based on analyses of lessons taught by preservice elementary teachers in their student-teaching semester. Although curriculum or curriculum use was not a variable considered in Sleep's identification of the tasks, specific strategies and problematic issues in these tasks serve as a useful tool to examine ways in which teachers steer their instruction toward the target mathematical idea and concept based on the existing resources and their plan.

METHODS

The setting of the study

Data were collected from 15 preservice teachers (PSTs) in a mathematics methods course. The five components of the capacity for productive resource use were intentionally used in the design of the mathematics methods course. For each written lesson (i.e., existing resources) drawn from a curriculum program (a curriculum program contains resources for teachers and students, such as the teacher's guides, student materials, and the implementation guide) as a context for exploring and discussing teaching practice and strategies, the PSTs were asked to complete surveys with open-ended questions about the lessons they read, before discussing them in class. The first three surveys included three questions: (1) mathematical goal of the lesson, (2) expected teacher role/actions, (3) components/aspects/parts/features of the lesson they think were important to use or keep in mind during instruction. The next three surveys included two additional questions: (4) expected student role/actions and (5) components, aspects, parts or features of the lesson that should be improved.

Field placement and lesson preparation

About half-way through the semester, each PST had an opportunity to teach a small group of third-grade students a 40-minute lesson on multiplication. The PSTs and I collaboratively prepared a plan focusing on relationships between multiplication facts. I drew on two activities from the same curriculum program that we had read and analysed a few times previously: (1) solving two related multiplication story problems and (2) writing a clue on each of multiplication cards to find the product. The two related story problems were: "1. *Oscar bought juice boxes that come in packages of 6. He bought 5 packs. How many juice boxes did he buy?*" and "2. *Pilar bought 8 packs* *of juice boxes. How many juice boxes did she buy?*" In solving the related problems, students are expected to find 5 packs of 6 juice boxes = 30 juice boxes first and then use this solution to figure out the next problem, that is, by adding 3 more packs of 6 to 30. The two activities (story problems and multiplication clue cards) were designed to support students' thinking about multiplication by asking them to relate one multiplication combination to another.

The PSTs recognized the potential of the two activities to support students' thinking about multiplication combinations. They already learned about strategies to support students' learning of basic facts including multiplication facts, such as using representations to attend to meaning and relationships. They decided to use the related story problems in the first activity as it was, and modified the multiplication clue cards activity to fit a small group interaction with a teacher. Based on the written lesson and previous discussions on teaching practice, the PSTs extensively discussed expected teacher moves in the lesson (two activities) in order to make a plan. The main ideas they highlighted about teacher moves in this lesson include:

- Use specific clear mathematical language (e.g., x-many groups of y-many things and the product)
- Ask probing questions (clarifying, justifying)
- Help students relate one multiplication problem to another
- Give clear directions (e.g., use the solution to the first problem to solve the next)
- Make a conclusion; ask students reflect on what they did in terms of the mathematical idea

Planning a lesson using the two activities, the PSTs explored potential student responses from the written lesson (guidance regarding the two activities) and additional student strategies and thinking along with a range of representations students might use. They identified questions from the written lesson that were important to use to probe students' thinking on the relationship between multiplication combinations, such as "*How did you know that you needed 3 more groups?*" and "*How can 5×4 help you figure out 6×4?*" Realizing that they needed many more specific questions, they also created a list of questions based on various anticipated student responses. For example, "*Why do you count by 3's?*" and "*We need 3 groups. How many are in each group?*" These questions were prepared to ask in various situations (e.g., when introducing expectations of the tasks, when students say, "I already know," when students do not relate one multiplication combination with another, etc.). Discussing the complexity of a range of multiplication combinations, they also picked and ordered those that they were going to use. They also prepared a list of additional multiplication combinations in case they needed. They carefully read the original activities and the lesson plan we created together in class. Although there were some PSTs who did not accurately identify the mathematical goal of the lesson, during the planning discussion they all became well aware of the goal, i.e., understanding and using relationship between multiplication combinations.

Data sources and analysis

The data gathered from this field placement include videotapes of teaching in small groups, photos of student solutions, the pre-survey on open-ended questions, and reflection papers on the teaching experience. PSTs' videos and written reflections on two particular episodes of their choice were crosschecked along with the specific strategies and issues of the tasks identified by Sleep (2012). Photos of student solutions were used to verify the PSTs' claims in their reflections. The pre-survey was used to determine the PSTs' sense-making of the lesson activities prior to whole group discussions, compared to collective sense-making and planning in the whole group.

RESULTS

Employing the construct of the capacity needed for productive resource use helped the PSTs attend to mathematical points, affordances, and limitations of the written lesson, and facilitated them to create a plan incorporating these ideas. The real challenge, however, arose when they enacted the plan to teach the lesson.

Identify the mathematical goal and important features of the lesson

The PSTs got better at identifying the mathematical point of a lesson/activity through the semester. For the two activities described above, 11 of 13 PSTs accurately elaborated the mathematical goal. It is important to mention that the activities given to them for analysis did not include overall information about the lesson, such as the list of the mathematics contents of the lesson. Given that, more PSTs recognized the main mathematical goal, compare to the first pre-survey on a lesson containing various information and descriptions about the mathematical goal and main content.

The pre-surveys also indicate that the PSTs attended to specific teacher moves and important features of the two activities and how these could support students' learning of the mathematics of the lesson, although the details, depth, and precision of their descriptions varied across the PSTs. Their common responses include: 'give clear directions—use the solution to the first problem to figure out the second problem,' 'ask students how to figure out the answer by using what is easier for them and explain their thinking,' and 'importance of the context of the problems given (the story problems).' Some PSTs also brought up the issues they recognized in the lesson. For example, they realized that they needed more teacher questions (clarifying and justifying) and additional potential student strategies besides those provided in the existing resources. Some suggested they use manipulatives for struggling students or give more specific instructions regarding the activities in instruction.

The PSTs' individual analyses of the lesson activities through specific survey prompts helped them prepare to make a plan together. These individual ideas were discussed during the planning time as described above in the methods section. The group discussions provided an opportunity for the PSTs to learn more about the lesson and what they should do to teach the lesson.

Steer the lesson toward the mathematical point

The PSTs prepared for the lesson with a clear mathematical goal, anticipated student strategies and thinking along with a range of representations, questions to ask in different situations, carefully chosen problems with purposefully selected numbers, specific teacher moves in the course of the lesson (introduction, first activity, second activity, and wrap-up) and ready-to-use materials. The entire preparation was done through whole group discussions. After the plan was made, they spent time on reading the plan and rehearsing what they were going to say and ask. During the teaching sessions, they focused on the problems (activities) and the intended mathematics. Nonetheless, the PSTs struggled to achieve the mathematical goal of the lesson to the desired level. The PSTs' teaching videos and written reflections revealed that there were multiple issues in their teaching to the mathematical point, although they made good moves in many instances. Sleep's elaboration of seven tasks for teaching toward the mathematical point helped identify the specific issues that the PSTs faced, which included:

- focusing students on the most relevant problems
- asking questions that engage students in mathematical reasoning
- lack of mathematical framing and narration
- spending more time on key ideas
- difficulty eliciting key ideas from students
- students not engaging in mathematical practices
- not capitalizing on opportunities to use meaning-focused language

Reflecting on their teaching sessions in the methods class, many PSTs mentioned that it was hard to make students engage with the intended mathematics because the students were not listening to each other after sharing what they did and because they claimed they just knew it. The management issues were certainly substantive. And yet, the bigger issue was that they did not, or were not able to, probe students' thinking sufficiently toward the intended mathematical goal.

The problems and examples were chosen carefully with particular numbers in order to spend time on the intended mathematics as Sleep indicated. Most students worked on the intended mathematics, relating one multiplication combination with another (use a known multiplication combination to find the product of another using the relationship between the two combinations) to solve problems. Probing these students' thinking carefully and focusing on the key idea (relationship between multiplication combinations) in their thinking, the PSTs could have engaged even those claiming they just knew it in discussion so that the students could spend more time on, and attend to, the meaning and relationship of multiplication combinations.

In written reflections, many PSTs confessed that they had a hard time probing students to think deeply about the relationship between multiplication combinations and using one combination to come up with the answer to another. Interestingly, while recognizing their struggles accurately in some instances, the PSTs thought they were doing okay in other instances although the videos and transcripts of those instances did not indicate that was the case. In fact, some reported instances of missed opportunities as examples of their good attempts.

IMPLICATIONS

Previously, I claimed that in order to use the existing resources productively, teachers need to clearly identify the mathematical points of lesson first, which seems to be an overarching task of the others (Kim, in press). Whereas identifying the mathematical goal is the first step toward productive resource use, the results of this study suggest that steering instruction toward the mathematical points of the lesson should be the aim/outcome, facilitated by the other four tasks. Before instruction, the PSTs were prepared regarding the four tasks; they identified the mathematical goal, affordances and limitations of the lesson, and made a plan to use important features of the lesson along with additional elements they created for their need. The last one—steering instruction toward the mathematical point—was the most difficult task when the PSTs used curriculum resources to teach mathematics. PSTs' planning describes the process in which they perceived and mobilized the resources before instruction. Mobilizing the resources during instruction, the PSTs had to steer instruction toward the mathematical point, which is part of PDC but requires the art of teaching. It seems that in-depth work on teaching practice is required for productive resource use.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). New York: Routledge.

Kim, O. K. (in press). Teacher capacity for productive use of existing resources.

Leshota, M., & Adler, J. (2018). Disaggregating a mathematics teacher's pedagogical design capacity. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 89–117). Cham: Springer.

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal, 49*(5), 935–970.

UNDERSTANDING MATHEMATICS TEACHERS' PEDAGOGICAL DESIGN CAPACITY IN CONTEXTS OF HIGH PRESCRIPTION

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Results from earlier research found that textbook use by selected secondary mathematics teachers in South Africa was largely tacit and not deliberate, making for un-intimate teacher-textbook relationships, and therefore low levels of teachers' pedagogical design capacity. The change from textbooks as primary curricular resources for teaching to daily scripted lesson plans begs a question about the kinds of relationships teachers forge with these plans. This article explores the affordances and constraints of a scripted lesson plan; and how one teacher mobilises these. Emerging results point to similarities between teacher-textbook interactions and teacher-scripted lesson plans interactions.

INTRODUCTION

For ease of implementation of a new curriculum in South Africa in 2012, annual teaching plans (ATPs) with week by week content coverage were introduced. In 2016, over concerns of persistent failure rates and poor curriculum coverage, the weekly plans were expanded to daily scripted lesson plans (SLPs) in one major province, mandating their use in poor performing schools. In the space of five years, teachers thus confronted a plethora of curricular resources for their teaching, each becoming more prescriptive of what was expected in a lesson and across the ATP. SLPs are presented to teachers as a finished product. They specify content, sequencing and duration of each lesson component. Relative to textbooks, they also come with reduced content elaboration. An ongoing larger research from which this article derives investigates how teachers interact with the SLPs. The study employs the notion of pedagogical design capacity (PDC) (Brown, 2009) to explore teachers' capacities to perceive and mobilise the affordances of the SLPs in effective ways.

Results from earlier study (Leshota, 2015) pointed to teachers' generally low PDCs with respect to textbooks. The change from textbooks to SLPs thus begs an important question that drives this article: what does PDC look like in contexts of high prescription. Studies of PDC are of interest in the South African context as well as internationally as PDC in mathematics education is under-researched. To date I have not found a study that addresses the issue of PDC in high prescription contexts. In this article, I present a case of one lesson by teacher Busi using an SLP to teach factoring of trinomials at Grade 9. This lesson was a pilot for the larger study to gain initial impressions of the new phenomenon of teachers using SLPs. I zoom in on the instructional explanations provided by both the SLP and Busi to illuminate processes

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by which teachers interpret the affordances of the SLPs and create deliberate and effective teaching strategies. In the necessarily brief literature review that follows, I focus on teachers' PDC and instructional explanations, and exclude attention to SLPs. My study is about PDC in the context of SLPs and not SLPs per se.

TEACHERS' PEDAGOGICAL DESIGN CAPACITY

Leshota and Adler (2018), proposed that a teacher's PDC is reflected in the kinds of *omissions* and/or *injections* that teachers make in their interaction with a resource. A combination of *robust injections* (enhancing the object) and *productive omissions* (not detracting from the object) indicates deliberate and participatory use and a high PDC. *Critical omissions* (critical to the object) or *distractive injections* (detracting from the object) point to tacit resource use and PDCs that are not high. As Brown (2009) posits, how much of the resource teachers use does not reflect their PDC since PDC "describes the manner and degree to which teachers create deliberate, productive designs that help accomplish their instructional goals" (p. 29). Remillard (2018) posits that familiar forms and structures of resources aid in the interpretive process for teachers, and shows how novel approaches and unfamiliar representations in curriculum resources might not be supportive of teachers' interpretive processes. The study seeks to unravel how teachers interpret the relatively unfamiliar forms but familiar content of the SLPs. While teachers' interpretive processes are influenced by their knowledge and goals, Leshota (2015) identified the *content* and the instructional *approach* as two main affordances of a resource for the teacher's practice. The sequencing of *presentation formats* (partitioning blocks) and learners' *performance expectations* (actions on tasks) in the resource illuminate the approach of a resource. The review above provides analytical constructs that I shall use for analysing the lesson.

INSTRUCTIONAL EXPLANATIONS IN MATHEMATICS PEDAGOGY

Explanations form an integral part of the teaching and learning of mathematics in the classroom. Leinhardt (2010) refers to explanations as powerful moments in teaching and posits that "the implicit assumptions need to be made explicit, connections between ideas need to be justified, representations need to be explicitly mapped"(p. 3). Similarly, Adler and Ronda (2015) argue for explanations as a key element of mathematics teaching. They place *explanatory talk*, as one of key mediational means that can bring the object of learning into focus for learners. In this article, the focus on explanations recognises their critical role in the classroom especially in contexts of SLPs that are characterised by reduced content elaboration.

THEORETICAL GROUNDING

This study is grounded in socio-cultural theory (Vygotsky, 1978) and the notion of mediated action, wherein, humans grow and develop through goal-directed use of tools. The SLPs are viewed as mediating artefacts for the teacher's practice that influence teachers' activity through their affordances/constraints. At the same time, the teacher may or may not mobilise affordances/constraints through her perceptions and decisions. My concern in this article are the instructional explanations afforded (or not) in the SLP and the teacher's mobilisation of these (or not) as she mediates mathematics to the learner in the lesson.

DATA COLLECTION

Busi was an alumni teacher of the project hosting the study who agreed to pilot the study in her classrooms. The course she attended exposed her to key elements of teaching including justifications as key elements in instructional explanations. Busi's lesson was audiotaped and photos of board-work taken. A post-observation interview followed a few months later to probe Busi's interactions with the SLP.

ANALYSIS

The analysis conducted was in two phases: firstly, determining the affordances and constraints of instructional explanations in the SLP lesson using presentation formats and performance expectations; then the related injections and omissions in Busi's lesson, complemented by her reflections in the interview.

Affordances and constraints of the SLP

A typical SLP runs between 30 and 35 minutes. Its outline includes the *topic; concepts and skills to be achieved*; *resources for teachers*; *prior knowledge*, and then lesson *components*. Five lesson components comprise: *mental maths* (5 min); *correction of homework* (5 min); *lesson presentation* (10–15 min) that comprises the heart of instructional explanations in the SLP; *classwork* (10–13 min); *consolidation* (2min), and teacher *reflections*. Figure 1 shows an allocation of 12 minutes for the present lesson.

Components	Time	TASKS/ACTIVITIES
LESSON PRESENTATION/ DEVELOPMENT	12 min	Revise the following with learners: Expand a) $2(x^2 + 4x + 8)$ b) $(x + 2)(x + 5)$

Complete the following activity with learners:
Factorise the following trinomial : $x^2 + 7x + 12$

How many terms are there in the above expression?	3 terms
What are the factors of the first term?	x x
List the pairs of factors of the last term / constant.	12 6 4 1 2 3
Which pair of factors when added gives the coefficient of the middle term?	4 3 $4 + 3 = 7$
Put both factors of the first term and last term into each of these brackets $(x \pm ?)(x \pm ?)$	$(x + 4)(x + 3)$ NB. The signs may not always be positive
Expand the factors to check if they give you the original trinomial	$x \times x + 3 \times x + 4 \times x + 4 \times 3$ $x^2 + 3x + 4x + 12$ $x^2 + 7x\ 12$

NB: Factorisation of an expression is the reverse of expanding an expression.

Figure 1. Lesson Presentation in the SLP

It begins with expanding two expressions followed by steps for finding factors. Sequencing both the *presentation formats* and learners' *performance expectations* identified the content and approach of the SLP, summarised in Table 1.

Presentation formats	Performance expectations	Comment
expansion of factors	- expand factors involving a common term: $2(x^2+4x+8)$ - expand factors involving a binomial: $(x+2)(x+5)$	rationale not explicit
factoring a trinomial	- identify factors of x^2 - identify factors of 12 that add up to 7 - determine factors of $x^2+7x+12$	no justification
expanding factors	expand factors, $(x+2)(x+3)$ to obtain $x^2+7x+12$	Equivalent forms

Table 1. sequencing of presentation formats and performance expectations in SLP

Table 1 reflects a distinct 'doing and undoing' (Mason, Graham, & Jonhston-Wilder, 2005) approach that iterates factoring and expanding. However, the rationale for including the expansion activity at the beginning is not explicit, nor is justification provided for the steps of the procedure. Thus, the SLP offers a plan that has potential for conceptual coherence, but lacks justification and rationale, what I refer to as the missing connective narrative. The expectation would be that Busi would perceive these affordances (conceptual coherence) and constraints (missing elaboration/justifications) and mobilise them effectively.

Analysis of Busi's mobilisation of the SLP lesson

Busi's instructional explanations comprised outlining steps of a procedure for factoring a trinomial, followed by three worked examples. The analysis identified the omissions and injections she made to the SLP lesson. She began her introduction as follows:

> Busi: when we are factorising a trinomial, the first thing we do is that we open up two brackets; the third term is the one that tells us whether the signs inside of the bracket are the same or not the same. So, if the third term is positive then that means the signs are the same…

The explanation does not provide justification of the steps just like in the SLP, implying that Busi was not able to perceive and then inject the missing connective narrative. This has implications for her PDC. When asked about this, she reflected that was how she had always taught factoring registering surprise for a need for justification. Busi's comment hints at broader issues of curriculum compliance versus teaching for understanding. Critical omissions in her mobilisation included emphatically telling learners not to expand the factors back to the original trinomial. This move undermined the conceptual coherence afforded in the SLP approach, as the doing and undoing of factoring and expanding as reverse processes was not made visible. Busi justified this move from her experience where learners confused the two tasks, leading her to separate the processes at all times. The injections she made included telling learners that the procedure she outlined worked only when the coefficient of x^2 was 1, and learners would need a new procedure when that changed, as well as working on additional and more varied examples which enhanced learners' skills with the procedure. Finally, Busi's comment about her use of the SLPs in general was that she used them only for examples, she did not use a textbook anymore, and preferred past examinations papers for planning her lessons. This reflects the multiple demands on her teaching decisions, and particularly a tension between teaching for understanding versus succeeding in examinations. In summary, the analysis here shows a mixture of robust injections and critical omissions, indicating some level of perception and mobilisation of affordances, but also a non-deliberate participation with the SLP. Busi's PDC while not high, is uneven.

DISCUSSION

The findings in this article point to a PDC that is not high, but also uneven for Busi. This is only one lesson and no conclusions can be drawn from it. Nonetheless, it alerts us to possible challenges on the ground: teachers not perceiving affordances and constraints of the SLP; deviating from the SLP and omitting critical aspects, yet also injecting productively; or utilising the SLPs selectively and effectively ignoring its other affordances and constraints, as Busi did. For contexts of high prescription, the findings suggest teacher-SLP interactions that might be tacit and not fully participatory pointing to similarities from study with textbooks. Scripting lessons this tightly was intended to be enabling for teachers, however, these suggest that the main challenge might lie with teachers' use of the resource. The results also illuminate the complex nature of this research while opening up avenues for further research.

References

Adler, J., & Ronda, E. (2015). A Framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237–254.

Brown, M. (2009). The teacher-tool relationship: theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). New York: Routledge.

Leinhardt, G. (2010). Introduction: Explaining instructional explanations. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 1–5): New York: Springer.

Leshota, M. (2015). *The relationship between textbook affordances and mathematics' teachers' pedagogical design capacity (PDC)*. Doctoral Thesis. University of the Witwatersrand. Johannesburg.

Leshota, M., & Adler, J. (2018). Disaggregating a mathematics teacher's pedagogical design capacity. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 89–117). Cham: Springer.

Mason, J., Graham, A., & Jonhston-Wilder, S. (2005). *Developing thinking in algebra.* London: Sage.

Remillard, J. T. (2018). Examining teachers' interactions with curriculum resource to uncover pedagogical design capacity. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 69–88). Cham: Springer.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

HOW ARE MATHEMATICIANS REPRESENTED IN CHINESE MATHEMATICS TEXTBOOKS?

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In this study, we investigate how mathematicians are represented in two series of primary and junior secondary school mathematics textbooks currently used in China, one used in Shanghai and the other by People's Education Press (PEP). The purpose of the study is to reveal the similarities and differences in the representation of mathematicians in the two series of textbooks, and explore the implications of the findings on mathematics textbook research and development. Our findings show that each series introduced both Chinese and non-Chinese mathematicians, though most of the mathematicians introduced were ancient mathematicians, and there is a higher level of consistency in the distribution of the introductions of mathematicians in both series in terms of mathematics contents and the structures of the chapters. However, the PEP mathematics textbooks introduced considerably more mathematicians than the Shanghai mathematics textbooks, and moreover, the PEP series presented a wider and, in our view, a better distribution of the introductions of mathematicians in terms of the grade levels, the ethnic origins, and the timeline of history.

INTRODUCTION AND BACKGROUND

Over the last few decades, mathematics textbook research has received rapidly increasing attention from researchers in different parts of the world (e.g., see Fan, Zhu, & Miao, 2013; Schubring & Fan, 2018). This development can be seen from the fact that, for example, ZDM published two special issues in 2013 and 2018, with the theme being "textbook research on mathematics education" and "recent advances in mathematics textbook research and development", respectively. The trend can be also observed from the holding of a series of International Conference on Mathematics Textbook Research and Development (ICMT) in the UK 2014, Brazil 2017, and Germany 2019.

In the general background of mathematics textbook research, there have been many related studies on the history of mathematics in mathematics textbooks (e.g., Erin, Bulut, & Bulut, 2015; Ju, Moon, & Song, 2016; Schorcht, 2018; Wang, Wang, & Hong, 2015), and parts of them indicate that the topic of mathematicians is an aspect of history of mathematics (e.g., Shen, Liu, & Wang, 2013; Wang, Wang, & Hong, 2015). However, there is rarely research on the representation of mathematicians in mathematics textbooks (Castaneda et al., 2019).

In this study, we aim to investigate how mathematicians are represented in the current Chinese school mathematics textbooks, by examining two series of mathematics textbooks, and explore the similarities and differences of the representation of

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mathematicians between the two series and their implications for the research and development of mathematics textbooks.

RESEARCH QUESTIONS

In the study, focusing on two series of Chinese primary and junior secondary school mathematics textbooks, one published and used in Shanghai and the other published by People's Education Press and used by many different parts of China, we intend to address the following questions:

- How are mathematicians represented in the mathematics textbooks?
- What are the similarities and differences on the representation of mathematicians between the two series of the mathematics textbooks?

METHODS AND PROCEDURE

Textbook selection

At present, there are two mathematics curriculum standards in the stage of compulsory education in China, i.e. the Shanghai Mathematics Curriculum Standards for Primary and Secondary Schools (Trial Version) and the Mathematics Curriculum Standards for Compulsory Education (2011 Edition). It should be noted that the Shanghai's curriculum standards is approved by the Chinese central government and implemented in virtually all the primary and secondary schools in the city of Shanghai. On the other hand, the Mathematics Curriculum Standards for Compulsory Education (2011 Edition), or simply the national mathematics curriculum, is implemented in all the regions except Shanghai in Chinese mainland. Both curriculum standards emphasize that the history of mathematics should be integrated into mathematics textbooks (Shanghai Municipal Education Commission, 2004; Ministry of Education of the People's Republic of China, 2011). The national curriculum standards particularly emphasized that students can understand better what the rigor of mathematics means from the work and spirit of mathematicians. To some extent, this reflects the importance of the introduction of mathematicians in Chinese mathematics textbooks.

As is well known, the stage of compulsory education in China consists of 9 years, from primary school to junior secondary school, but the school system in Shanghai is different from other parts of China, which follow the national curriculum standards. There are 5 years in primary education, followed by 4 years in junior secondary education in Shanghai, while in all the other parts of the country, the primary education consists of 6 years and the junior secondary education consists of 3 years, so it is more comparable if we select the whole stage of compulsory education.

In this study, to investigate how primary and junior secondary school mathematics textbooks in China represent mathematicians, as mentioned earlier, we selected the two series of mathematics textbooks.

For Shanghai, the mathematics textbooks used were developed in accordance with the Shanghai's curriculum standards. Due to some reason, they were published by two publishers, the first and second grade of the mathematics textbooks were published by Juvenile and Children's Publishing House, while all the remaining seven grades were published by the Shanghai Education Publishing House.

People's Education Press (PEP) was established by the Ministry of Education of the People's Republic of China, and specialises in developing school textbooks, and the series of mathematics textbooks we selected in this study were developed according to the national mathematics curriculum.

Process

After selecting the textbooks, we used content analysis method and examined all the textbooks to address the research questions mentioned above with focus on the following aspects.

- Regarding how mathematicians are represented in the mathematics textbooks, we focused on how the topics of mathematicians are distributed in the textbooks in terms of the grade level, mathematical content, the structure of chapters in the textbooks, the nationality and the periods of time.
- Regarding the similarities and differences on the representation of mathematicians between the two series of textbooks, we further examined the textbooks from the perspective of comparison in terms of the distribution as mentioned above.

More specifically, focusing on the representation of mathematicians in the selected textbooks, we looked at the distribution in the following aspects. Firstly, consistent with the Chinese national curriculum standards, we divided mathematical contents into the three broad areas: number and algebra, shape and geometry, and probability and statistics. Secondly, we analysed each chapter in the textbooks in terms of the introduction, main texts, examples, exercises and reading materials. Finally, we categorised the nationalities or ethnic origins of the mathematicians introduced in the textbooks into Chinese and not Chinese in terms of the periods of time, i.e., ancient time (before 1840), modern time (1840–Semptember 1949) and contemporary time (October 1949–present), which is based on the commonly used division of the timelines of the Chinese history in China.

FINDINGS AND DISCUSSIONS

The distributions of the introductions of mathematicians in both series of the selected textbooks across the different grade levels are reported in Table 1.

Grade level	1	2	3	4	5	6	7	8	9	Total
Shanghai	0	0	0	0	2	8	5	18	3	36
PEP	0	2	1	8	6	7	21	22	12	79

Table 1. Distribution of the introductions of mathematicians across different grade levels in the selected textbooks

It can be seen from Table 1 that the number of mathematicians mentioned in Shanghai's mathematics textbooks is less than half of the number in textbooks published by People's Education Press. Moreover, the introduction of mathematicians in the Shanghai mathematics textbook is heavily concentrated in the junior high school stage, especially in the eighth grade. In contrast, the distribution in the PEP textbooks is much more widely spread, and better in our view, over both primary and junior secondary stages, though still tilted towards the junior high school stage, which we think is understandable.

Mathematics contents	Shanghai	PEP
Number and algebra	16	42
Shape and geometry	20	35
Probability and statistics	0	2

Table 2. Distribution of the introductions of mathematicians in terms of mathematics contents

As is shown in Table 2, although the PEP mathematics textbooks introduced more mathematicians than Shanghai's textbooks, the introduction of mathematicians in both series of mathematics textbooks is concentrated in (1) number and algebra and (2) shape and geometry, and the numbers in the two areas are highly consistent.

In contrast, the introduction of mathematicians in probability and statistics is little, especially in the Shanghai series. This is consistent with the fact that the content of probability and statistics only takes up a small percentage in the compulsory education stage in China.

Structure of chapters	Introduction	Main texts	Examples	Exercises	Reading materials
Shanghai	4	2	0	0	30
PEP	0	11	0	6	62

Table 3. Distribution of the introductions of mathematicians in terms of structure

From Table 3, we can see that the introduction of mathematicians in both series of mathematics textbooks is predominantly found in reading materials and little is found in the remaining parts, and in particular, no examples mentioned mathematicians. It is understandable that students can learn more about mathematicians by reading materials than other ways.

Table 4 shows the distribution of the introductions of Chinese and non-Chinese mathematicians in the two series.

As we can see from Table 4, most mathematicians introduced in both series of the mathematics textbooks belong to ancient times; and only a very small number of mathematicians introduced in the textbooks belong to the modern and contemporary times, which is particularly the case in the Shanghai textbooks.

Again, the PEP textbooks clearly have a more widely spread distribution of the introduction of mathematicians both in terms of their ethnic origins and in terms of timelines of history.

Series	Nationality	Ancient	Morden	Contemporary
Shanghai	Chinese	16	0	1
	Not Chinese	18	1	0
PEP	Chinese	23	2	5
	Not Chinese	43	4	2

Table 4. Distribution of the introductions of mathematicians in terms of ethnic origins in timeline of history

CONCLUDING REMARKS

Based on the findings reported above, it is easy to see that both series of the Shanghai mathematics textbooks and the PEP mathematics textbooks paid some attention to the introduction of mathematicians, which is particularly clear in the junior secondary school stage as compared to the primary school stage. We think this treatment is related to the content of mathematics for students to learn in the two different stages and suggest that the primary mathematics textbooks introduce more mathematicians' stories.

Regarding mathematics contents and the structure of chapters, the introduction of mathematicians is concentrated on the topics of (1) number and algebra, (2) shape and geometry and (3) reading materials, respectively. The former is consistent with the distribution of mathematics contents in mathematics curriculum standards at the compulsory education stage, while a possible reason for the latter is that students can learn more about mathematicians by reading materials than other ways. Thus, we recommend that the rest of the structure of chapters should be more balanced.

We think it is commendable that each of the Shanghai and PEP mathematics textbooks introduced both Chinese and non-Chinese mathematicians, though most of the mathematicians introduced are ancient mathematicians, which in our view is understandable as the knowledge covered in the school textbooks is mainly basic knowledge in mathematics that mankind has known or discovered since ancient times. But maybe the introduction of modern and contemporary mathematicians in mathematics textbooks make students more interested in mathematics.

Finally, from a comparative perspective, the mathematics textbook series by the PEP presented a wider, a more balanced and hence, in our view, a better distribution of the introductions of mathematicians in terms of grade levels, ethnic origins and the timeline of history. It would be interesting to find out what the reasons are behind the different treatments as mentioned earlier, which is beyond the scope of this study and deserves further investigation.

References

Castaneda, A., Aguilar, M., Gómez-Blancarte, A., Romo-Vázquez, A., Lezama-Andalón, J., & Miranda-Viramontes, I. (2019). Representations of mathematicians in lower secondary mathematics textbooks. *Eurasia Journal of Mathematics, Science and Technology Eductaion, 15*(7). https://doi.org/10.29333/ejmste/106056.

Eren, M., Bulut, M. & Bulut, N. (2015). A content analysis study about the usage of history of mathematics in textbooks in Turkey. *Eurasia Journal of Mathematics, Science and Technology Eductaion, 11*(1), 53–62.

Fan, L., Zhu, Y. & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education, 45*(5), 633–646. https://doi.org/10.1007/s11858-013-0539-x.

Fan, L., Xiong, B., Zhao, D., Niu, W. (2018). How is cultural influence manifested in the formation of mathematics textbooks? A comparative case study of resource book series between Shanghai and England. *ZDM Mathematics Education, 50*(5), 787–799. https://doi.org/10.1007/s11858-018-0976-7.

Ju, M., Moon, R., & Song, J. (2016). History of mathematics in Korean mathematics textbooks: Implication for using ethnomathematics in culturally diverse school. *International Journal of Science and Mathematics Education*, *14*(7), 1321–1338. https://doi.org/10.1007/s10763-015-9647-0.

Ministry of Education of the People's Republic of China (2011). *Mathematics Curriculum Standards for Compulsory Education (2011 Edition)* (pp. 63). Beijing: Beijing Normal University Publishing Group. (In Chinese)

Shanghai Municipal Education Commission (2004). *Shanghai Mathematics Curriculum Standard for Primary and Secondary Schools (Trial Version)* (pp. 93). Shanghai: Shanghai Education Publishing House. (In Chinese)

Schorcht S. (2018). History of mathematics in German mathematics textbooks. In K. M. Clark et al. (Eds.), *Mathematics, education and history* (pp. 143–162). Cham: Springer. https://doi.org/10.1007/978-3-319-73924-3_8.

Schubring, G., & Fan, L. (2018). Recent advances in mathematics textbook research and development: An overview. *ZDM Mathematics Education, 50*(5), 765–771. https://doi.org/10.1007/s11858-018-0979-4.

Shen, C., Liu, D., & Wang, X. (2013). "Simple solid figures" of high school mathematics textbooks in China, Singapore, America and France. *Journal of Mathematics Education, 22*(4), 30–33. (In Chinese)

Wang, J., Wang, X., & Hong, Y. (2015). A comparative study on mathematical culture in Chinese, French and American high school textbooks. E*xploring Education Development,* (20), 28–32. https://doi.org/10.14121/j.cnki.1008-3855.2015.20.007. (In Chinese)

ON THE INTRODUCTION OF VECTORS IN GERMAN TEXTBOOKS FOR UPPER SECONDARY SCHOOL

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In this paper, we present a praxeological analysis of (introductory) tasks on vectors in German textbooks for secondary schools in the state North Rhine-Westphalia. We motivate our research interest from our work in transition research from school to university. The Anthropological Theory of the Didactic (ATD) has already proven to be suitable for textbook analysis in other studies and was chosen as a framework for our research. With the reconstruction of a praxeological model from the textbooks, an initial analysis with a minimal institutional bias was possible. Among the results is a model for the analysis of the introduction of vectors in textbooks (with a focus on tasks), and evidence for differences and similarities between the analysed textbooks.

INTRODUCTION

Problems and issues regarding students' transition from school to university have been observed for some time now. This is a general phenomenon (Briggs, Clark, & Hall, 2012) and has a specific shape in mathematics. Internationally, several supportive measures were realized in order to ease the transition processes for students. Some are designed to support freshmen parallel to their studies in their first term, others aim to close gaps prior to the beginning of studies at university (e. g. Bausch et al., 2014; Gleason et al., 2010). Within the studiVEMINT project, we have developed an online course designed for self-regulated learning which is freely available to all students in Germany. Among the covered topics is an introduction to vectors. At the start of designing the course, we wondered, what prior knowledge on vectors freshmen bring along. The results of a short study revealed that students bring many different concept views and that German textbooks do not follow a coherent approach with regard to the introduction of vectors either (Mai, Feudel, & Biehler, 2017), although vectors are an important topic in German upper secondary schools.

Whilst our initial study focused on what freshmen have learned on vectors, we wanted to shift our focus towards what students are offered to learn on vectors at school level in more detail. Since textbooks are an "essential part of teaching" (González-Martín, Giraldo, & Souto, 2013), the study presented in this paper is designed to reconstruct the knowledge taught on vectors by the textbooks. We intended to use a methodology for our analysis which is not restricted to the specifics of our topic of interest (c. f. Ronda & Adler, 2017). Coming from a transitional point of view, we also intended to use a methodology that is sensitive on the institutional contexts of presented mathematics. In conclusion, we chose the Anthropological Theory of the Didactics as a framework for our analysis as described in the following section (Bosch & Gascón, 2014).

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THEORETICAL BACKGROUND AND LITERATURE REVIEW

Literature on the topic of vectors in a school related sense is scarce. Vectors often only occur as elements of vector spaces and in connection with linear algebra (e. g. Dorier, 2003). Research dedicated specifically to vectors seems to investigate mostly specific aspects like representation forms (Watson, Spyrou, & Tall, 2003), students' ideas about lines in vector contexts (Wittmann, 2003) or the dot product as an operation with vectors (Donevska-Todorova, 2015). At upper secondary school level, concepts of vectors in geometrical contexts (especially, translations and equivalence classes of arrows), in arithmetical contexts (*n*-tuples), and as objects with shared structural properties are relevant (Henn & Filler, 2015).

From an Anthropological Theory of the Didactic (ATD) point of view "Doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, are considered as human activities taking place in institutional settings" (Bosch & Gascón, 2014, p. 68). A key resource from the framework of ATD is the notion of praxeologies, which consist of a praxis block and a logos block:

> The praxis block is made of "types of tasks" and a set of "techniques" (considering this term in a broad sense of "ways of doing") to carry out some of the tasks of the given type (those in the "scope" of the technique). The logos block contains two levels of description and justification of the praxis. The first level is called a "technology," using here the etymological sense of "discourse" (logos) of the technique (technè). The second level is simply called the "theory" and its main function is to provide a basis and support of the technological discourse. (Bosch & Gascón, 2014, p. 68)

Praxeologies can have different scopes. A praxeology which includes only one type of task (and an according set of techniques), is called point praxeology. Praxeologies including two or more types of tasks which refer to a shared piece of logos are called local praxeologies. Despite the emphasize of human activities, ATD is also applied in textbook analysis (e. g. González-Martín et al., 2013; Wijayanti & Winslow, 2018). Textbooks are institutional artefacts involved in processes of teaching and learning. Among the advantages of analysing textbooks with the help of ATD is comparability between the textbooks because the model for an analysis can be reconstructed from the textbooks themselves. Thus, the model fits for all selected textbooks.

METHODOLOGY

Research questions

Based on the described theoretical background, we will investigate two research questions. **RQ1:** Which task-oriented praxeological model can be reconstructed by the analysis of the (chosen) textbooks? **RQ2:** Which differences and similarities can be identified by comparing the textbooks based on the praxeological model.

Data selection

We chose three German textbooks relevant in the state of North Rhine-Westphalia: Neue Wege (NW) (Körner, Legemüller, Schmidt, & Zacharias, 2015), Lambacher Schweizer (LS) (Brandt et al., 2014) and Elemente der Mathematik (Elem) (Griesel, Gundlach, Postel, & Suhr, 2014). Thus, all textbooks share the same curricular context and have been published in 2014 or 2015. For this study, we narrowed the analysed content down to the introduction of vectors. We considered all tasks starting from the introduction page of vectors up to the beginning of the next section (addition of vectors etc.). All tasks related to the length/norm of vectors were disregarded in the analysis in favour of comparability because LS only introduces the length/norm of a vector in a later section.

Data analysis

In the long term, we want to reconstruct a wholesome praxeological model from the textbooks. This paper aims to approach the praxeological modelling starting from the praxis blocks by classifying types of task. **Step 1:** All tasks on vectors between the introduction page and the next book section have been identified and documented in each textbook. **Step 2:** We synthetised elementary types of tasks (point praxeologies) from the textbooks like "Finding the corresponding vector between two given points." Usually, any identified type of task can be modelled again as constituted of two or more underlying, more specific types of tasks. ATD itself does not state at which level of detail the modelling process should come to a hold. We modelled up to a level of detail at which still an explicit reference to vectors is necessary. For example, the type of task mentioned above would break down into types of task involving arithmetic of real numbers to calculate the vector components. Thus, we stopped at that level. Step 2 is completed when each textbook task can be attributed to at least one type of task and no more elementary types of task can be found. **Step 3:** Formerly identified point praxeologies were bundled into groups which share a mutual piece of. Thereby, local praxeologies were formed. Doing as described results in a task-oriented praxeological model reconstructed from the textbooks. Every task in the textbooks can be attributed to at least one elementary type of task in reference to the praxeological model. We coded all tasks defined in step 1 according to our reconstructed model. In this process, multiple codes have been attributed to the same task in some cases. If a task was divided into subtasks, we coded each subtask individually.

RESULTS

Regarding **RQ1**, our analysis resulted in the identification of a number of point praxeologies and their aggregation into local praxeologies as follows. (P1) Point and vector relations: (P1.1) Given two points, find the connecting vector, (P1.2) given a vector and its endpoint, find the starting point, (P1.3) given a vector and its starting point, find the endpoint, (P1.4) finding the opposite vector of a tuple. (P2) Depiction of vectors: (P2.1) draw a vector tuple in the plane as an arrow (P2.2) draw a vector tuple in the space as an arrow, (P2.3) recognize arrows belonging to the same vector. (P3)

Argumentation and exploration: (P3.1) what is the opposite vector of the opposite vector, (P3.2) reasoning with prior knowledge about geometry, (P3.3) explore a phenomenon. (P4) Modelling with vectors: (P4.1) transpose a context into a mathematical situation, (P4.2) transpose mathematical situation into a context, (P4.3) reflect on modelling assumptions, (P4.4) introduce a coordinate system into a given situation.

By a translation the vector $\vec{V} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$ maps the point P to the point Q. Find the coordinates of the missing point Q respectively P.

a) $P(12|-8|25)$ c) $P(-1|-3|-7)$
b) $Q(-6|15|17)$ d) $Q(q|q-5|3q+2)$

Given the location vector of the point P: Find P and draw the point P together with the location vector and appropriate auxillary lines into a coordinate system.

a) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ c) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ d) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Figure 1. Two translated exemplary tasks from Elem (left) and LS (right). The left task's subtasks were coded as (P1.2) or respectively (P1.3). The right task's subtasks were coded as (P6.2) and (P6.3).

(P5) Compare two vectors. (P6) Position vectors: (P6.1) find the position vector to a given point, (P6.2) find the point to a given position vector, (P6.3) draw an arrow from the origin to the designated point of the position vector in space. Afterwards, we could attribute the praxeologies to all task selected for analysis as shown in figure 2. In the analysis included were eight tasks (consisting of 13 subtasks) from NW, 12 tasks (consisting of 43 subtasks) from LS, and 8 tasks (consisting of 20 subtasks) from Elem.

	P1.1	P1.2	P1.3	P1.4	P2.1	P2.2	P2.3	P3.1	P3.2	P3.3	P4.1	P4.2	P4.3	P4.4	P5	P6.1	P6.2	P6.3
NW	7 (54%)	1 (8%)	5 (38%)	-	1 (8%)	1 (8%)	1 (8%)	-	2 (15%)	2 (15%)	-	1 (8%)	-	2 (15%)	-	-	-	-
LS	18 (42%)	3 (7%)	3 (7%)	-	6 (14%)	2 (5%)	-	1 (2%)	9 (21%)	1 (2%)	1 (2%)	1 (2%)	1 (2%)	2 (5%)	5 (12%)	-	6 (14%)	6 (14%)
Elem	2 (10%)	3 (15%)	7 (35%)	6 (30%)	-	1 (5%)	2 (10%)	-	-	-	-	-	-	-	1 (5%)	1 (5%)	-	-

Figure 2. The absolute number of codes for each point praxeology and their occurrence relative to the number of coded subtasks for each textbook (NW: 13, LS: 43, Elem: 20).

Based on the model from RQ1, we can discuss similarities and differences between the textbooks, leading to **RQ2**. A clear focus seems to be on (P1) for each book. LS focuses on (P1.1), calculating a vector out of two given points. Elem focuses on (P1.3), finding the endpoint of a given vector together with its starting point. NW covers both. (P1.4), finding the opposite vector of a tuple, only occurs in the Elem textbook. (P5) is the only point praxeology that could not be aggregated into a local praxeology.

From figure 2 it becomes evident that all three textbooks only commonly share (P1.1), (P1.2), (P1.3) and (P2.2). In each analyzed textbook section different praxeologies are missing. The number of missing praxeologies is 7 for NW, 3 for LS and 9 for Elem. The number of "praxeology-gaps" in figure 2 is evidence for different approaches and foci on introducing and teaching vectors in the textbooks. Many praxeologies only occur once or twice in a textbook task. How many times a point praxeology was identified relates to its emphasis in the respective textbook section. Only (P1.1) and (P1.3) occur at least 5 times in at least to different textbooks. (P1.1) and (P1.3) are part of the same local praxeology (P1). They share common logos which is connected to arithmetic representations of points and vectors (as tuples). In general, (P1) seems to be the most important local praxeology. The impression arises that calculating with vectors (tuples) and points (also as tuples) is the backbone of the introduction of vectors. Almost every other point praxeology occurs rarely in the textbook sections. A cautious interpretation could be that they are included to support (P1) rather than building up to another local praxeology apart from (P1).

CONCLUSION AND FUTURE RESEARCH POTENTIAL

Although (P1) is a dominant local praxeology in each textbook, it was made evident that each textbook uses a different approach towards the introduction of vectors. This is surprising, considering that the analysis was only conducted on elementary introductive tasks for vectors. A more common and coherent approach could have been expected. Future research should investigate the effect of these differences on students' learning. Praxeologies apart from (P1) seem to be considered as arbitrary and interchangeable. More textbooks could be included for the analysis to investigate this. The reconstructed praxeological model would eventually become more detailed and it could be possible to identify groups of textbooks sharing similar approaches towards teaching vector introduction (on a task-level). Further, the question arises, how many tasks should represent a praxeology to create a significant opportunity to learn about it.

We did not present (detailed) results on the logos part of the textbook praxeologies. It seems worthwhile to further investigate the connection between the praxis and logos parts in the textbooks. It becomes evident that tasks with an arithmetical-centered logos are favored throughout the textbooks. Also, the nature and structure of the presented logos is of interest for us. Therefore, we are currently working on developing a praxeological reference model from our institutional point of view in order to analyze the logos parts and especially to systematically identify gaps and implicit logos pieces. Vectors occur as tuples, arrows (as equivalence classes), and translations. From our results above, an emphasis on tuples could be revealed, demoting arrows and translations to a less dominant position. Yet, research on the understanding of vectors and related concepts seems to ascribe more meaning to non-algebraic/non-arithmetic representations.

References

Bausch, I., Biehler, R., Bruder, R., Fischer, P. R., Hochmuth, R., Koepf, W., … Wassong, T. (2014). *Mathematische Vor- und Brückenkurse*. Wiesbaden: Springer Fachmedien.

Bosch, M., & Gascón, J. (2014). Introduction to the Anthropological Theory of the Didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 67–82). Cham: Springer.

Brandt, D., Giersemehl, I., Greulich, D., Herd, E., Jörgens, T., Jürgensen-Engl, T., … Zinser, M. (2014). *Lambacher Schweizer Mathematik Einführungsphase Nordrhein-Westphalen* (1st ed.). Stuttgart: Ernst Klett.

Briggs, A. R. J., Clark, J., & Hall, I. (2012). Building bridges: Understanding student transition to university. *Quality in Higher Education*, *18*(1), 3–21.

Donevska-Todorova, A. (2015). Conceptual understanding of dot product of vectors in a dynamic geometry environment. *Electronic Journal of Mathematics & Technology*, *9*(3), 192–209.

Dorier, J.-L. (2003). Teaching linear algebra at university. In L. Tatsien (Ed.), *Proceedings of the International Congress of Mathematicians* (Vol. III, pp. 875–884). Beijing: Higher Education Press.

Gleason, J., Boykin, K., Johnson, P., Bowen, L., Whitaker, K., Micu, C., … Slappey, C. (2010). Integrated Engineering Math-Based Summer Bridge Program for Student Retention. *Advances in Engineering Education*, *2*(2), 1–17.

González-Martín, A. S., Giraldo, V., & Souto, A. M. (2013). The introduction of real numbers in secondary education: An institutional analysis of textbooks. *Research in Mathematics Education*, *15*(3), 230–248.

Griesel, H., Gundlach, A., Postel, H., & Suhr, F. (Eds.). (2014). *Elemente der Mathematik: Nordrhein-Westfalen, Einführungsphase*. Braunschweig: Westermann Schroedel Diesterweg Schöningh Winklers.

Henn, H.-W., & Filler, A. (2015). *Didaktik der Analytischen Geometrie und Linearen Algebra*. Berlin: Springer.

Körner, H., Legemüller, A., Schmidt, G., & Zacharias, M. (Eds.). (2015). *Mathematik Neue Wege Arbeitsbuch für Gymnasien: Einführungsphase*. Braunschweig: Westermann Schroedel Diesterweg Schöningh Winklers.

Mai, T., Feudel, F., & Biehler, R. (2017). A vector is a line segment between two points? - Students' concept definitions of a vector during the transition from school to university. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1–5, 2017)*. Dublin: DCU Institute of Education & ERME.

Ronda, E., & Adler, J. (2017). Mining mathematics in textbook lessons. *International Journal of Science and Mathematics Education*, *15*(6), 1097–1114.

Watson, A., Spyrou, P., & Tall, D. (2003). The relationship between physical embodiment and mathematical symbolism: The concept of vector. *The Mediterranean Journal of Mathematics Education*, *1*(2), 73–97.

Wijayanti, D., & Winslow, C. (2018). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *Journal of Research in Mathematics Education*, *6*(3), 307.

Wittmann, G. (2003). *Schülerkonzepte zur Analytischen Geometrie: Mathematikhistorische, epistemologische und empirische Untersuchungen*. Hildesheim: Franzbecker.

ANALYSIS OF TEXTBOOKS IN THREE LATIN COUNTRIES: RESOLUTION OF EQUATIONS AND INEQUALITIES

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In this article, we examine and compare how the procedures for solving one-variable linear equations and inequalities are presented in the narrative of Brazilian, Portuguese, and Spanish textbooks. We analyze the main resources used, the relationships between the procedures of solving equations and inequalities, and the reasoning-and-proving opportunities in the exposition of content. Results include the fact that all books use scales as a resource to illustrate some property in order to solve a specific equation, however, this feature is merely used as an empirical example, and not as a reasoning-and-proving opportunity.

INTRODUCTION

Textbooks play a key role in classrooms considering that “textbooks have a strong influence in mathematics teaching and learning” (Ponte & Marques, 2007). Such an influence can be observed because textbooks give teachers and students access to knowledge, and “textbooks have their power in providing an organized sequence of ideas and information to structured teaching and learning, which guide reader’s understanding, thinking, and feeling” (Sosniak & Perlman, 1990 as cited in Fan, Zhu, & Miao, 2013, p. 635).

This work is an ongoing investigation which will integrate the first author’s master dissertation on the contents of one-variable linear equations and inequalities. Analyzing some Brazilian books, we realized that these contents are presented in a disconnected way and that the properties of equality and inequality used in solving equations and inequalities appear only with empirical justifications. However, it is important that the development of mathematical argumentation be carried out in the classroom, since, according to Stylianides (2009, p. 259), “it is unreasonable to expect that students will develop proficiency in RP [Reasoning-and-Proving] unless this activity receives systematic attention in mathematics instruction”.

In addition, we believe that, since there is a deep relationship between equation and inequality solving procedures, if the equation solving properties are worked out developing RP, it will be only natural to relate these two topics exploring conjectures and asking some important questions about this relationship.

In this article, we also analyse the procedures for solving equations and inequalities and the opportunities for the development of RP in Portuguese and Spanish books. “These opportunities are characterized by statements or exercises that take the reasoning-and-proving process as an explicit object of attention or reflection” (Otten et

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 239–244). Paderborn: Universitätsbibliothek Paderborn.

al., 2014). However, in this article, we only analyze the opportunities presented in the content exposition.

METHODOLOGY

As indicated earlier, the objective of this article is to analyze the presentation of equation and inequality solving procedures found in textbooks from three different countries. Inspired by Ruthven (2014, p. 26) “the textbooks in widespread use in England tend to present mathematics as a set of unrelated rules and facts”, we checked whether the books present relations between the analyzed contents. In addition, we looked at whether the books provide opportunities for students to have the chance to read and reflect on RP, since these exposures in the text may be the basis for students to apply later with RP (Hong & Choi, 2018, p. 89). In analyzing the RP in the narratives, we are inspired by Stylianides (2009), who uses the hyphenated term for the fact that the process of proving is much greater than the proof itself, consisting of empirical exploration, conjecture, generalization, refining, explaining and providing proofs.

We will make a documentary analysis of the narrative of the textbook, using the ideas of Otten et al. (2014), “within each sampled lesson, we identified any expository mathematical statements such as theorems, postulates, properties, formulas, or identities. Mathematical definitions were not coded, but worked examples that included a prompt for RP were. The statements included in the analysis were then coded for their mathematical statement and their justification-type”. We will address the following questions: (1) What resources are used to present the procedures for solving equations and inequalities? (2) What are the relationships between the procedures for solving equations and inequalities? (3) What types of statements are found in textbooks regarding the resolution of equations and inequalities? (4) What types of justifications are used in textbooks for the procedures for solving equations and inequalities? To answer these questions, we use the framework for content exposure analysis given inspired by Hong & Choi (2018) and Otten et al. (2014).

Hong & Choi (2018, p. 85) defines a general statement as a class of mathematical situations or objects and a particular statement as a specific mathematical object or situation. About justification types, they state that deductive justifications are defined when the textbook presents a logical argument based on definitions, postulates, or previously established results to support or prove a mathematical statement, while empirical justifications are defined when the textbook presents an example, or a set of examples that confirm the assertion given (Otten et al., 2014, p. 62).

In our analysis, we used teacher's editions of books adopted in public schools in different countries. The chosen Brazilian books were the most used in our country: Bianchini (2015) and Dante (2016), which aroused our curiosity in verifying how these contents were presented in other countries. Hence, our natural choices were Costa & Rodrigues (2013a, 2013b) from Portugal, and Jiménez, Albero, & Cañas (2015) and Jiménez, Albero, González, & Cañas (2015) from Spain, regarding the language’s similarities.

TEXTBOOKS ANALYSIS

We present here a summary of the analysis of the chosen books focusing on the contents of solving equations and inequalities and seek to answer the questions pre-established in our methodology. The analyses were made by the two authors independently, and afterwards they were compared to observe the similarities and differences. It is important to point out that only the first appearance of these contents was analyzed in the textbooks.

In each Brazilian book the contents of equations and inequalities appear in different chapters, one following the other, and are taught in the same year, 7th year. In the Portuguese and Spanish books these contents appear in different books: equations in the 7th year and 1st eso and inequalities in the 9th year and 4th eso respectively; in these books the content of inequalities appears followed by the chapter on the set of real numbers and all sets of solutions are presented in interval form.

Introduction to procedures

The two-plates scale is used as a resource in solving equations in both Brazilian books, with the difference in the order of their presentation: Bianchini (2015) starts with examples of a balanced scale, then uses the properties of equality; while Dante (2016), starts with inverse operations and afterwards, explores examples of a balanced scale. The books of Portugal and Spain begin the resolution of equations with mental math examples; then through a particular case using a balanced scale (see Figure 1), the properties are displayed. The Spanish book also presents representations similar to scales and algebraic bricks, but only in specific examples.

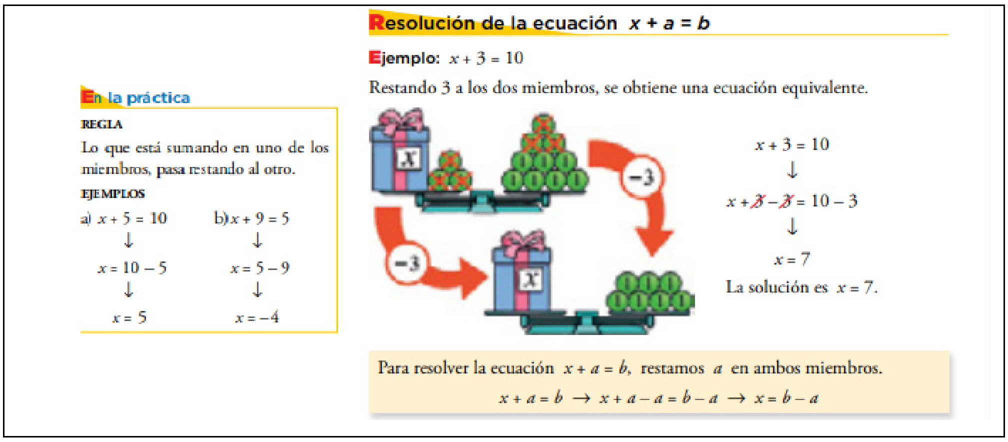


Figure 1. Additive property using the scale (Jiménez et al., 2015, p. 180)

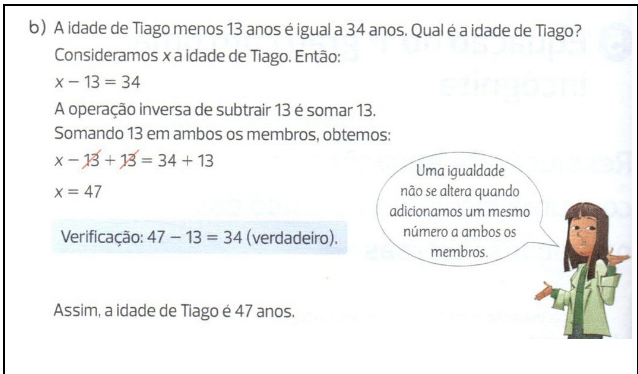
In the content of inequalities, the two-plates scale is only used to illustrate an example, and in only one of the analyzed books (Bianchini, 2015), which then uses the properties of inequality to solve inequalities. Since Dante (2016) hardly explores the content of inequalities, he only explores the mental math in a simple example: $2x > 6$. The Portuguese book mentions that there is a set of rules and then solves a specific example of an inequality using properties which are never made explicit. In the Spanish book the resolution of inequalities is introduced by the graphic representation of a line. First-degree inequalities are briefly addressed, along with second-degree inequalities. The algebraic resolution is presented directly by suggesting: “To solve a one-variable linear inequality, we proceed as if it were an equation with the following exception: if we multiply or divide by a negative number, the inequality changes its direction” (Jiménez et al., 2015, p. 68).

Comparison between solving equations and inequalities

Few relationships are presented between the contents of equations and inequalities. In Dante (2016), for example, no relation is mentioned. In Bianchini (2015), the content of equations is quoted in two statements in the chapter of inequalities: "just as in equations, inequalities also have two members" (p. 124) and "in solving one-variable linear inequalities, we can use the same process we studied in solving equations" (p. 128), however, the relations of difference between the resolutions of the equations and inequalities were not presented. The Portuguese book presents the statement: "as we have seen for the equations, there is a set of rules applicable in the resolution of inequalities" (Costa & Rodrigues, 2013, p. 65); this is the only comparison of the two contents. In the Spanish book the only relation that appears between the two contents is the last sentence quoted in the previous section.

Mathematical statements, justifications and reasoning-and-proving

In all the analysed books we realize that the properties of both equations and inequalities presented are general statements (a total of 23), followed, preceded, or presented simultaneously by specific examples, which are considered particular statements (illustrated in Figure1). We note that only the Spanish book presents the general statements in symbolic language (see Figure 1), all the others present them in their mother language. As the specific examples are the only "justifications" presented, we consider that all books present only empirical justifications. In Figure 2 there is an example in which the author presents the additive property of equations justified only with the validation of an example.


b) A idade de Tiago menos 13 anos é igual a 34 anos. Qual é a idade de Tiago?
Consideramos x a idade de Tiago. Então:
$x - 13 = 34$
A operação inversa de subtrair 13 é somar 13.
Somando 13 em ambos os membros, obtemos:
$x - \cancel{13} + \cancel{13} = 34 + 13$
$x = 47$
Verificação: 47 − 13 = 34 (verdadeiro).
Assim, a idade de Tiago é 47 anos.

b) The age of Tiago minus 13 is equal to 34 years. What is the age of Tiago? We consider x the age of Tiago. Then:

$x - 13 = 34$

The inverse of subtracting 13 is to add 13. Adding 13 to both sides, we get:

$x - \cancel{13} + \cancel{13} = 34 + 13$

$x = 47$

An equality does not change when we add the same number to both sides.

Verification: 47 – 13 = 34 (true).

Therefore, the age of Tiago is 47 years.

Figure 2. General Statement with Empirical Justification (Dante, 2016, p. 127)

In spite of the fact that the empirical logical linkage is presented in many examples, in a clear way, comparing with the representations made with scales or other illustrations, RP appears only empirically in the justifications presented in the analysed books.

Other elements, such as the conjecturing, generalizing, and providing evidence in mathematics, are not presented in these books.

Suggestions

The scale is a good resource not only to illustrate the properties of equality and inequality (with positive coefficients) empirically, as presented in several textbooks, but also as a way of generalizing and comparing the contents of equations and inequalities. We list some ideas that could be explored and added to the content exposition of textbooks. Due to the limited space, our suggestion involves the additive property of equality, but it can also be adapted to other properties: (1) Illustrate with several empirical examples how to obtain a balanced scale. Use also decimal and fractional masses and different shapes with the same mass; (2) Ask questions such as: If a scale is in equilibrium, what is the conclusion about the mass on each plate? If we have two objects of the same mass, one on each plate of a scale, does it remain in equilibrium?; (3) Illustrate similarly to (1) in order to let students observe what happens on the scale when equal amounts are added to each of the two plates and then create conjectures about what they are observing. Also illustrate the inverse operation: remove equal masses from each of the two plates in a balanced scale. (4) Ask questions, for example: If a scale is balanced and we add to each plate an object of the same mass, does it remain in equilibrium? And if we remove from each plate of a balanced scale the same mass does it remain in equilibrium? 5) Generalize with masses named a, b and c, for example. Ask a final question: And if the masses are not balanced? We suggest repeating items 1 through 5 making the necessary adjustments.

FINAL CONSIDERATIONS

In this paper, we analysed textbooks from three different countries regarding the presentation of the equations and inequalities resolution procedure and the RP opportunities in the exposition.

The most used resource in the books is the two-plate scale which, although being an excellent precursor of conjectures, was not used in this way; it appeared only in specific examples, justifying the general statements in an empirical way.

An important issue in teaching inequalities is the relationship that can be made to equations, since the procedures for resolution are very similar, although having two points that distinguish them: solution (uniqueness in equations and non-uniqueness in inequalities) and the multiplicative property (when we multiply--or divide--both members of an inequality by a negative number, it inverts its direction). However, the question of the uniqueness of the solution did not appear in the analysed books, and the relations between the main properties used to solve equations and inequalities were presented incompletely.

We note that in all books, although the properties (of equality and inequality) are classified as general statements, all justifications were examples, classified as empirical justifications. However, it is important that, in addition to empirical examples, opportunities to conjecture, generalize, or provide evidence be presented, which are part of the RP process proposed by Stylianides (2009). Thus, RP could be further developed in the content exposition of these textbooks, since it is through these examples that students can have a basis for solving exercises involving RP (Hong & Choi, 2018). Complementary proposals that contemplate the opportunity for RP are presented in the Suggestions section.

Aiming at the importance of studies in the field, in future works we intend to analyse the exercises of these books to verify if they provide the development of the RP and propose alternatives of development of the RP in the classroom, using the two-scale scale as well as algebraic tiles.

References

Bianchini, E. (2015). *Matemática. 7º Ano*. São Paulo, SP: Moderna.

Costa, B., & Rodrigues, E. (2013a). *Novo Espaço 7ºano (parte 2)*. Porto: Porto.

Costa, B., & Rodrigues, E. (2013b). *Novo Espaço 9ºano (parte 2)*. Porto: Porto.

Dante, Luiz R. (2016). *Projeto Teláris: Matemática* 7º ano. São Paulo, SP: Ática.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook Research in Mathematics Education: Development Status and Directions. *ZDM Mathematics Education, 45,* 633–646.

Hong, D. S. & Choi, K. M. (2018). Reasoning and proving opportunities in textbooks: A comparative analysis. *International Journal of Research in Education and Science, 4*(1), 82–97. DOI:10.21890/ijres.382937

Jiménez, J. C., Albero, G. I., & Cañas, R. C. (2015). *Matemáticas 1. ESO.* Madrid: Anaya Educación.

Jiménez, J. C., Albero, G. I., González, M. J. O., & Cañas, R. C. (2015). *Matemáticas orientadas a las Enseñanzas Académicas 4 ESO.* Madrid: Anaya Educación.

Otten, S., Males, L., & Gilbertson, N. (2014). The introduction of proof in secondary geometry textbooks. *International Journal of Educational Research, 64,* 107–118.

Ponte, J. P., & Marques, S. (2007). Proportion in school mathematics textbooks: A comparative study.

Ruthven, K. (2014). The Textbook is dead: long live to the textbook. In K. Jones, L. Fan, G. Howson, & C. Bokhove (Eds), *Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT-2014*). Southampton: University of Southampton.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11(4),* 258–288.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*, 253–295.

EXAMINING PROPORTIONAL REASONING IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

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Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks. The aim of this study is to examine the extent to which Turkish, Singaporean and Canadian middle school mathematics textbooks provide students opportunities to experience proportional reasoning. The results show that the selected textbooks in Singapore provide the strongest opportunity for proportional reasoning.

INTRODUCTION

Proportional thinking is an important skill at middle school level; it is a long-term development, and has some important steps (Lamon, 2012). Proportional thinking is linked with many of the main topics of mathematics (such as similarity and trigonometry), and also with some of the concepts that we often encounter in everyday life (such as density and velocity) (Lesh, Post, & Behr, 1988). The National Council of Teachers of Mathematics (NCTM) (2000) has highlighted that proportional reasoning should be "an integrative theme in the middle grades mathematics program" (p. 212). However, many students do not show their proportional thinking skills at the desired level. An important reason for this is the poorly designed teaching design (Labato & Ellis, 2010).

Studies on mathematics textbooks show that textbooks play a very important role in the learning and teaching process (Rezat, 2006). Because of this important role of the textbook, it can be considered as a mediator between the intended and applied curriculum (Schmidt, McKnight, & Raizen, 1997), and therefore textbooks are important research areas in terms of mathematics education.

Both the importance of textbooks and proportional thinking in mathematics education have led to different studies in this field. A group of studies have focused on tasks which examine the types of proportional problems and solutions and cognitive demands (Bayazıt, 2012; Holzrichter, 2016). Other studies have addressed the main components of proportional thinking more extensively (Shield & Dole, 2013; Ahl, 2016). When reviewing the literature, it was found that there are no studies on the development of proportional thinking in textbooks. For this reason, this study could provide valuable information about textbook structure and sequencing, in terms of proportional reasoning. On the other hand, teachers may get the chance to re-organise their education plans when they realise the strengths and weaknesses of textbooks.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 245–250). Paderborn: Universitätsbibliothek Paderborn.

The aim of this research is to compare the extent to which selected Turkish, Canadian and Singaporean textbooks provide opportunities for the development of proportional reasoning. The topics of ratio, proportion and rate will therefore be examined in this study.

Shifts as a Model for Proportional Reasoning Development

Proportional reasoning is an important skill that requires a long process of development. In this process, students need to make multiple shifts in terms of being "adept at forming ratios, reasoning with proportions, and creating and understanding rates" (Labato & Ellis, 2010, p.61).

Labato and Ellis (2010) presented four significant transition models for the development of proportional thinking. Table 1 represents these shifts and their main ideas.

Shifts	Main idea
1) From one quantity to two	Make a transition from focusing on only one quantity to realising two quantities
2) From additive to multiplicative comparisons	Make a transition from making an additive comparison to forming a ratio between two quantities
3) From composed-unit strategies to multiplicative comparisons	Make a transition from using only composed-unit strategies to making and using multiplicative comparisons
4) From iterating a composed unit to creating many equivalent ratios	Make a transition from developing a few "easy" equivalent ratios to creating a set of many equivalent ratios

Table 1. Proportional reasoning shifts and their main ideas

The first shift, which involves the reasoning of two quantities, is the focus of the relationship between two variables, rather than a single variable. Ellis (2007) suggested that posing problems can motivate students to isolate an attribute in quantitative situations. Harel et al. (1994) asked students about the "orangey"-ness of different sized cups filled with the same amount of orange juice. He stated that some students answered, "The big cup is more orangey, because it has more orange then little cup." This example shows that these students were reasoning with only one variable (size of cup). Since proportional thinking involves the relationship of two different variables, students must first notice the relationship between these variables. For this reason, firstly students have to realise the relation of the two variables.

The second shift is about the difference between additive and multiplicative reasoning. Proportional reasoning involves a multiplicative relation. Ellis (2007), in his study involving a number of interconnected wheels, showed that the small wheel turned three times whilst the big wheel turned two times. When asked about the relationship between the number of turns of the wheels, a student responded by saying that in an additive relationship "the number of turns of the small wheel is greater than one from the big wheel". This interpretation involves additive reasoning. In order to interpret the proportionality, students must realise the difference between multiplicative and additive relationships.

Shift 3 involves using the effective multiplicative relationship between variables (e.g. 1:3=3:?, 3:1=3⇒ 3x3=9), rather than creating equivalent ratios based on iterating and partitioning a composed unit in order to create a family of equivalent ratios (e.g. 1:3=3:?, 1:3⇒2:6⇒ 3:9) (Lesh, Post, & Behr, 1988).

Shift 4 involves creating infinite sets of equivalent ratios rather than iterating easily composed units (e.g. doubling and halving numbers). Here, rate definition is important. Rate is defined by Thompson (1994) as a set of many equivalent ratios. This definition allows the students to gain a more sophisticated understanding of proportional reasoning (Lobato & Ellis, 2010).

These transitions that Lobato and Ellis (2010) have created for the development of proportional thinking provide a model for the examination of textbooks. Since these transitions involve the development of students' thoughts, it is important to create textbook chapters in accordance with this development.

METHOD

Turkish, Canadian and Singaporean textbooks were selected for this study. The reason for this choice is primarily the success of Singaporean and Canadian students in international examinations, such as The Program for International Student Assessment (PISA). The 2015 PISA results saw 15-year-old Singaporean students ranked 1st, Canadian students ranked 9th, and Turkish students ranked 40th, in terms of mathematics performance. Furthermore, the results of the 2015 TIMMS for Grade 8 students on the topics of "ratio, proportion and percent" emphasised the difference between the achievements of students from these countries. In response to questions within this study, 82% of Singaporean students and 55% of Canadian (Ontario) students got the correct answers, compared with just 34% of Turkish students.

A new syllabus series in use in Singapore was chosen for analysis in this study. Since Canada consists of many states and has access to numerous textbooks, the Nelson Education series used in Ontario was used in this study. The textbooks of Gizem and Mega Publishing have been selected for Turkey. All of these textbooks have been passed through an approval process by their respective governments for use in classrooms in these countries. The ratio, proportion and rate topics were selected for analysis. For this reason, Grade 5, 6, 7 and 8 textbooks were selected for Singapore, Grade 6, 7 and 8 textbooks were selected for Canada, and Grade 6 and 7 textbooks were selected from Turkey.

The units determined for the analysis are divided into four main blocks. These are narratives, tasks, examples and representations. In this study, the narrative block consists of any definitions and explanations related to the topic. A task is identified as any question with no solution. Examples involve any question which has a solution. Finally, representation contains figures, pictures and graphs related to the topics. These content structures were used for rating the shifts. If a shift is not presented by any structure, its rating is none; if a shift is presented by one structure, its rating is weak; for two or three structures the rating is medium; and for four structures the rating is strong.

RESULTS

Shift 2 is rated as weak for the Grade 7 Turkish textbook. It gives only one task, which is: "1 litre of lemon juice and 4 litres of water are used for 5 litres of lemonade. How can we compare lemon juice and water quantities?"

There is another example of the rating of Shift 2 from the Grade 5 Singaporean textbook. It gives an example which asks, "How many ways can we compare the number of lemons and number of pears?", which refers to the difference between multiplicative and additive thinking (p.120). The example features a picture model representing the numbers of lemons and pears. Two different comparisons are given in the solution. These are: "There are 6 more pears than lemons/there are 6 fewer lemons than pears"; and "There are twice as many pears as lemons/there are half as many lemons as pears". The example continues with a transition, which states that "the number of lemons to the number of pears is 6 to 12". Finally, the definition of ratio is given as follows: "Such comparison of numbers or quantities is called a ratio". However, there is no task related to the difference between multiplicative and additive thinking. For the reason that it contains three main structures, Shift 2 is represented in the selected Singaporean textbooks as having a medium rating.

The Canadian Grade 7 textbook provided a strong opportunity for Shift 3. There are examples, tasks and templates for using the scale factor. Also, Shift 3 is supported with a narrative which states that "you can multiply or divide each term in a ratio by to get the equivalent terms in another ratio; it can be a whole number or a decimal" (p.42).

The findings from the analysis of all of the textbook series are summarised in Table 2. From Table 2, the Singaporean textbooks represent Shift 3 and Shift 4 strongly, Shift 1 is represented weakly, and Shift 2 is rated as medium. The Canadian textbook series supports Shift 3 strongly, Shift 1 is rated medium, and for Shift 2 and Shift 4 there is no evidence. In the Turkish textbook series, Shift 4 is rated strong, Shift 3 is rated medium, and Shift 1 and Shift 2 are rated as weak.

	Singapore	Canada	Turkey
Shift 1	Weak	Medium	Weak
Shift 2	Medium	None	Weak
Shift 3	Strong	Strong	Medium
Shift 4	Strong	None	Strong

Table 2. Rating of textbook series

One of the limitations encountered in the study is that it does not give the same topic at the same grade levels in each of the three countries. For example, the topics of direct and inverse proportion and linear relationships are discussed in the Canadian textbooks after Grade 8. This situation could be one reason why no clues about Shift 4 were found in the Grade 6, 7 and 8 textbooks.

CONCLUSION

Proportional thinking is a skill that sees a bridge in middle school mathematics, and its development takes time. Students have to understand these important transitions in this process. Considering the place of textbooks in mathematics education, it is important that textbooks include these transitions in terms of proportional thinking. In this study, it has been concluded that the selected series of Singapore textbooks supported these transitions in the strongest way, while the Canadian and Turkish textbooks supported them more weakly. These results provide important feedback to both textbook authors and curriculum deciders. On the other hand, determining the level of students in proportional thinking will guide teachers to organise their lessons.

References

Ahl, L. M. (2016). Research findings' impact on the representation of proportional reasoning in Swedish mathematics textbooks. *REDIMAT – Journal of Research in Mathematics Education*, *5*(2), 180–204.

Bayazit, I. (2013). Quality of the tasks in the new Turkish elementary mathematics textbooks: The case of proportional reasoning. *International Journal of Science and Mathematics Education*, *11*(3), 651–682.

Bilen, O. (2017). *Ortaokul matematik 7 ders kitabı*. Ankara: Gizem.

Ching, T. P., & Jitan, L. (2008). *New Syllabus Primary Mathematics Textbook 5*. Singapore: Shinglee.

Ching, T. P., & Jitan, L. (2008). *New Syllabus Primary Mathematics Textbook 6*. Singapore: Shinglee.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education,* 194–229.

Güven, D. (2017). *Ortaokul matematik 6 ders kitabı*. Ankara: Mega.

Harel, G., Behr, M., Lesh, R., & Post, T. (1994). Invariance of ratio: the case of children's anticipatory scheme for constancy of taste. *Journal for Research in Mathematics Education,* 324–345.

Holzrichter, R. (2016). *Proportional reasoning in middle level mathematics textbooks*. (master thesis). University of Northern Iowa.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. New York, NY: Routledge.

Lesh, R., Post, T., & Behr, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.) *Number concepts and operations in the middle grades* (pp. 93–118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Lobato, J., & Ellis, A. B. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8.* Reston, VA: National Council of Teachers of Mathematics.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International Results in Mathematics*. Retrieved from Boston College, TIMSS & PIRLS International Study Center website: http://timssandpirls.bc.edu/timss2015/international-results/

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA.

OECD (2019). *Mathematics performance (PISA)*, (indicator). https://doi.org/10.1787/04711c74-en (Accessed on 12 January 2019)

Rezat, S. (2006). A model of textbook use. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková, *Proceedings of the of the 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 409–416). Prague: PME

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *Splintered vision: An investigation of US mathematics and science education*. Norwel, MA: Kluwer.

Shield, M. J, & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics, 82*(2), 183–199.

Small, M., & Kestell, M. L. (Eds.), (2006). *Nelson Mathematics 6*. Canada: Thomson Nelson.

Small, M., & Kestell, M. L., & Zimmer, D. (2005). *Nelson Mathematics 7*. Canada: Thomson Nelson.

Small, M., & Kestell, M. L., & Zimmer, D. (2006). *Nelson Mathematics 8*. Canada: Thomson Nelson.

Yeo, J., Teh, K. S., Loh, C. Y., Chow, I., Neo, C. M., Liew, J., & Ong, C. H. (2013). *New Syllabus Mathematics 1* (7th ed.). Singapore: Shinglee.

Yeo, J., Teh, K. S., Loh, C. Y.,e Chow, I., Neo, C. M., Liew, J., & Ong, C. H. (2014). *New Syllabus Mathematics 2* (7th ed.). Singapore: Shinglee.

TEXTBOOK CONTENT IN USE: MANUAL AND MACHINE CODING

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We investigate how mathematical content in a textbook is taken up by instructors, as they plan and teach their lessons, and by students, who participate in those lessons. We first identify mathematical competencies afforded by the textbook applying manual and automatic coding of the textbook's raw content that uses natural language processing techniques. Next, we analyse textbook usage reported by users collected via periodic surveys and automatic real-time textbook viewing data with the goal of determining whether textbook features lead to the activation of competencies by the students.

INTRODUCTION

Within the array of resources for teaching and learning, the textbook continues to be the most prevalent one for instructors and students. With new technological developments, textbook formats have been changing from paper to digital and open source formats, including sophisticated tools such as computing cells, annotation tools, and powerful search engines, easing mass access at relatively low cost. Importantly, open source textbooks never expire or go out-of-print and can be distributed at no cost to students, making them practically fully accessible. In countries in which post-secondary education costs are high wide accessibility contributes to eliminating financial barriers to education. However, the full potential of these textbooks can only be understood through empirical studies of how students and instructors actually use these enhanced textbooks. The study we report here is part of the large project that seeks to describe how instructors and students use open-source, technologically enhanced university mathematics textbooks. We specifically want to see the extent to which the competencies espoused in the textbooks are talked about by the students using those textbooks which may speak indirectly about how instructors enforce those competencies.

THEORETICAL BACKGROUND

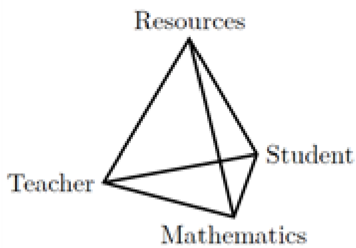

Figure 1: Didactical tetrahedron

The didactical tetrahedron (Rezat & Strässer, 2012) helps in understanding the pivotal role of resource use in teaching (Figure 1). In the base of the tetrahedron are elements of the instructional triangle, a definition of instruction as the interaction among the instructor, the students, and the content. In this model, resources are an interdependent element that modifies such interaction. In addition, the model allows researchers to attend simultaneously to students' use of the resources to learn mathematics (Students-Mathematics-Resources) and to the

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 251–256). Paderborn: Universitätsbibliothek Paderborn.

interdependent way in which instructors and students interact with the resources. Although initial conceptualizations of use have been proposed about textbook readers, ours is a purposeful attempt to investigate the use of textbooks for teaching and learning in real time.

Niss and Højgaard (2011) defined mathematical competency as "a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (p. 49). Their framework can be thought of an intersection of various sets of competencies believed to be necessary for students who seek a STEM degree. For this study, we constructed a mathematical competence framework inspired by the Kompetencer Og Matematiklæring (KOM, Competencies and mathematics learning) model (Niss, 2003), by Maaß's work (2006), and by the joint position on mathematical skills for STEM set by the National Council of Supervisors of Mathematics and National Council of Teachers of Mathematics (2018). We focus on the following eight competencies: (1) *Mathematical Thinking*, (2) *Posing and Solving Mathematical Problems,* (3) *Mathematical Modelling,* (4) *Mathematical Reasoning,* (5) *Representing Mathematical Entities,* (6) *Handling Mathematical Symbols and Formalisms,* (7) *Communicating in, with, and about Mathematics,* and (8) *Using of Aids and Tools*. Several elements are used to describe each competency as a whole. For example, the *Mathematical Thinking and Acting* competency includes: i) posing questions that have a mathematical characteristic; ii) understanding and handling the scope and limitations of a given concept; iii) extending the scope of a concept; and iv) distinguishing between different kinds of mathematical statements (conditioned assertions, 'if-then', assumptions, definitions, theorems, conjectures, and cases).

CONTEXT AND METHODS

The data we analyzed for this investigation have been drawn from a more extensive study that investigates students' and instructors' use of open-source dynamic textbooks in calculus, linear algebra, and abstract algebra courses. *Linear Algebra* (Beezer, 2019), is an open source, dynamic textbook written using PreTeXt, an authoring markup language designed specifically by our development team to produce interactive online textbooks. This textbook includes the following features: theorems, proofs, definitions, exercises (and solutions), examples, Sage cells (interactive computational cells), and Reading Questions. In particular, Reading Questions, are interactive areas that invite students to write answers directly into the textbook. As students respond to the questions, the instructor receives them, in real time. PreTeXt captures the structure of textbooks to ease conversion to multiple other formats (https://pretextbook.org/).

To analyze the textbook features, we applied manual and automatic coding, using 25% of the textbook as training data, with nine sections randomly chosen out of 39. The manual coding constituted the first phase of analysis and was based on the competency framework. We coded the textbook by identifying which competencies were being activated within each textbook feature. The competencies are not mutually exclusive;

concurrent activation is possible as are multiple activations of the same competency within the same textbook feature: e.g., the reasoning competency was activated in several parts of the same proof. However, we record whether activation of the competency occurs or not, and not the frequency or quality of its activation. To establish intercoder agreement, three individuals coded two sections in the *Dimension*[1] subsection of the Vector Spaces section. This subsection included 44 items and were coded by textbook features. The inter-rater agreement was 91% (agreement in 40 out of 44 items). Discrepancies resulted from similarity in definitions for the competencies; those were clarified.

Once coding reached a sufficient amount of textbook content, we used the data as the training dataset for the text classification models. The dataset needed to contain several hundred of labeled text documents to ensure a reasonable degree of accuracy for the model. The second phase of the analysis included statistical cross-validation methods to train the model iteratively using automatic coding using natural language processing techniques that derived a set of competencies for the rest of the textbook. Once the model was mature enough, we run the model for the rest of the textbook to tag feature section content with different mathematical competencies. The model itself continuously evolved and improved reaching higher accuracy.

To complement the study of the textbook, we analyzed survey responses filled out by over 100 students who were taking linear algebra courses with the HTML version of the textbook. The students were in six different courses taught by instructors located in different states; the data were collected during the spring of 2018. We used topic modeling to analyze these responses in a scalable and efficient manner. GENSIM library was selected to conduct Latent Dirichlet Allocation (LDA, Blei, Ng, & Jordan, 2003) topic modeling on each survey question. The LDA algorithm aims to map the documents to a predefined number of topics; each topic is mapped to a number to keywords that help describe it. Our script repeatedly applied the topic modeling algorithm based on a range of topic number we specified beforehand. We picked the model with the highest evaluation score as the final model. For each question, we generated three tables: the number of documents for each topic, representative documents for each topic, and the dominant topics for each document. The outputs were generally very interpretable, with meaningful differences between topics. One particularly useful feature of LDA topic modeling is that it identifies the most representative student responses for each question. With these results, we identified the sections of the textbook that students have used most heavily as well as the specific ways they interacted with the textbook. These results were further compared with the actual usage data generated from HTML textbook server log. Critical pieces of information to uncover from the server log include the frequencies of access by students for each section of the textbook during different timeframes, the number of students accessing each chapter, as well as the general usage trends exhibited by the

[1] https://books.aimath.org/fcla/section-D.html

class throughout the semester. While, the amount of coded textbook samples and student responses was relatively small, the findings provide sufficient confirmation that the processes for analyzing the textbooks and the student responses can yield important patterns of use.

FINDINGS

We present first the findings regarding the competencies that appear to be afforded by the textbook features followed by the textbook features that seem to be most used by the students, according to the LDA topic modelling analysis. The frequency of coded features by competencies are presented in Table 1.

Features (#)	Mathematical Thinking	Posing & Solving Math Problems	Mathematical Modelling	Mathematical Reasoning	Representing Math Entities	Handling Symbols & Formalism	Communicating in, with, about Math	Using Aids and Tools
Definition (n=34)	32	0	0	0	29	28	4	0
Example (n=65)	58	3	1	25	50	48	29	0
Sage Cell (n=25)	3	22	0	0	1	4	1	22
Theorem (n=78)	71	0	0	1	70	38	33	0
Reading Questions (n=30)	21	22	0	14	11	15	18	0
Exercise (n=147)	72	123	2	42	87	107	72	0
Proof (n=77)	68	0	0	66	68	14	31	0
Total	325	170	3	148	316	254	188	22

Table 1. Frequencies of competencies coded by feature in the textbook.

There are primarily two ways to interpret Table 1. First (feature-oriented), we focus on a competency and check how it is enacted across different textbook features. By calculating the proportion for each feature (the active level) given the fixed competency[2]. Active level can be seen as a compromise between the proportion of any particular textbook feature within our sample and the proportion of the fixed competency within the given textbook feature. These two proportions determine active level. If a feature is not highly present in the textbook but a particular competency is highly reflected in such feature, such combination will not likely be deemed as prominent. Second (competency-oriented), we focus on one textbook feature and check how different competencies are reflected within that feature. For example, there are 34 definitions in total and 32 of them reflect mathematical thinking, the percentage we assign is 94%. Any percentage above 50% is deemed "very high," between 12.5% and 50% is deemed "high," and under 12.5% is deemed low.

Competencies afforded in textbook features

From a feature-oriented perspective, we found that the feature *Exercises* addressed a wide variety of mathematical competencies: seven out of eight have their highest active level in the exercises. *Examples* closely followed and achieved prominence (see Table 1) in six out of eight competencies. Overall, *Exercises* and *Examples*, addressed

[2] We deem a feature prominent if the proportion is above 15%: 100 percent out of 7 features ~15%.

a high variety of mathematical competencies. In addition, five out of eight competencies were found in at least three to four textbook features, which suggests that the textbook balances and presents different competencies in a variety of features. We found that the modelling competency rarely appeared in the textbook. Two features, *Definitions* and *Sage Cells*, seem to address a narrower spectrum of competencies; the former, achieved virtually no prominence across all eight competencies; the latter focuses exclusively on the eighth competency (*Making Use of Aids and Tools*). These frequencies might not be surprising, as we could expect that different features be relevant for different competencies. As we document how students interact with particular textbook features, we will be able to anticipate which competencies are they more likely to activate. From a competency-oriented perspective, *Mathematical Thinking*, *Representing Mathematical Entities*, and *Handling Mathematical Symbols and Formalisms,* are prevalent across many textbook features, each dominating in at least four to five out of seven features. *Mathematical Reasoning* and *Communicating in, with, and about Mathematics* are moderately prevalent, each achieved high active level (between 13% and 50%) in four to five out of seven textbook features. We also found that the competencies *Mathematical Modeling*, *Making Use of Aids and Tools*, and *Posing and Solving Mathematical Problems* competencies are relatively rare to find across textbook features, with less than three textbook features emphasizing the development of these competencies.

Textbook features most used by students

One survey question asked the students to describe what they did while using the textbook. Among the representative responses using Examples was frequent. Students seem to use the Examples as a starting point while studying a section: "*While reading the textbook I will mostly ignore large blocks of words or explanations and move straight to the examples*." Examples were also the main source for creating their personal documents for the course: "*I will do the same example multiple times until it's very known.*" The textbook's homework exercises, including the embedded solutions, were also identified as a main textbook feature, used while preparing for a particular section. The LDA topic modeling method revealed multiples references to the use of exercises and solutions by the students: "*The solutions to the exercises at the end of the sections were the most useful for me*." Therefore, based on the findings of the previous section, we can assume that students will activate a wide variety of mathematical competencies even though they seem to engage mostly with only two textbook features.

DISCUSSION

We sought to explore what competencies are afforded by the textbook features and to what degree the students interacted with the textbook features. The results suggest an association between the mathematical content offered by the textbook—in the form of mathematical competencies—and the competencies that the students will likely be asked to bring into action and possibly improve on while interacting with particular textbook features. If students are interacting primarily with the *Examples* and *Exercises*, then they will likely have to activate and advance various competencies: mathematical thinking skills, use of mathematical entities and symbols, and posing and solving mathematical problems. We also found competencies that were less prominent and some that were not very common in the textbook: this textbook does not offer many opportunities for students to advance their mathematical modelling skills (e.g., features that would ask students to interpret mathematical results in an extra-mathematical context). Likewise, *Making Use of Aids and Tools* was rarely found in the textbook (only afforded by one textbook feature, Sage). Finally, the *Posing and Solving Mathematical Problems* competency was highly prominent within *Exercises,* but it was relatively rare in the features of the textbook.

The LDA topic modelling we used to analyze students' responses in our survey was facilitated by incorporating a summary of features the students emphasize most. The LDA method identified the most representative responses. However, to interpret, derive conclusions, and create the summary, we needed to read and understand what these selected responses were highlighting regarding the use of the textbook features. A future step towards the analysis of the textbooks is to expand the competency framework to include finer categories, in order to obtain more nuanced patterns.

References

Beezer, R. (2019). *First course in linear algebra*. Gig Harbour, WA: Congruent Press. Retrieved from https://books.aimath.org/.

Blei, D. M., Ng, A. Y., & Jordan, M. I. (2003). Latent Dirichlet allocation. *Journal of Machine Learning Research, 3*(Jan), 993–1022.

Maaß, K. (2006). What are modeling competencies? *ZDM Mathematics Education, 38*(2), 113–142.

National Council of Supervisors of Mathematics, & National Council of Teachers of Mathematics. (2018). *Building STEM education on a sound mathematical foundation (Joint position)*. Retrieved from https://www.nctm.org/Standards-and-Positions/Position-Statements/Building-STEM-Education-on-a-Sound-Mathematical-Foundation/:

Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis & S. Papastavridis (Eds.), *3rd Mediterranean Conference on Mathematical Education* (pp. 115–124). Athens, Greece: Hellenic Mathematical Society and Cyprus Mathematical Society.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teadhing and learning in Denmark*. Roskilde: Roskilde University.

Rezat, S., & Strässer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM Mathematics Education, 44*, 641–651. https://doi.org/10.1007/s11858-012-0448-4

HOW JAPANESE TEACHERS USE MATHEMATICS TEXTBOOKS FOR "KYOZAI-KENKYU": CHARACTERIZING THEIR DIFFERENT USES BY PARADIDACTIC PRAXEOLOGIES

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"Kyozai-kenkyu" is an essential part of Japanese lesson study and involves task design for classroom lessons. Most Japanese teachers refer to textbooks as a main resource for "kyozai-kenkyu". However, there are different kinds of uses according to teachers' perspectives towards the knowledge at stake. In this study, we examined teachers' paradidactic activities related to "kyozai-kenkyu" using mathematics textbooks. For this purpose, some notions of Anthropological Theory of the Didactic (ATD) are adopted by analysing "kyozai-kenkyu" about the knowledge of fractional numbers in primary schools. As a result, we identified three types of paradidactic praxeology (in-textbook, with-textbook, and beyond-textbook) which are related to different aspects of didactic transposition processes.

INTRODUCTION

Kyozai-kenkyu (教材研究) is not only an essential part of lesson study (cf. Shimizu, 2014), but also an important aspect of daily classroom practice for Japanese teachers (Fujii, 2015). Literally, the word *kyozai* means teaching materials or subject matter, and *kenkyu* refers to research or study. In a *kyozai-kenkyu* practice, task design is involved and textbooks are often used as a main resource for it. Since Japanese lesson study has been internationally well known, the word *kyozai-kenkyu* has been also used in international contexts (Watanabe et al., 2008). Watanabe et al. (2008) presents a general description of *kyozai-kenkyu*; however, it is also important to note that there are different types of *kyozai-kenkyu* as well as different usage of textbooks according to teachers' perspectives towards the knowledge at stake. The purpose of this paper is to characterise Japanese primary teachers' *kyozai-kenkyu* using mathematics textbooks. We characterise such differences by means of theoretical constructs from the Anthropological Theory of the Didactic (ATD).

Generally speaking, the textbooks are very influential for Japanese teachers' practice. On the other hand, just as there must be various types of teacher, so the uses of textbooks must also be diverse. In fact, especially among elementary school teachers, not all of them are necessarily confident in mathematics. The textbook companies usually publish supplementary resources for a textbook, such as teachers' editions and teaching guides (including commentaries of all contents, syllabus of lessons sequence, and some lesson plans). These materials are helpful for some teachers, allowing them to meet the national curriculum standard (called *Course of Study*) if they teach the lesson content exactly as described. Because the Ministry of Education must approve

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all textbooks based on *Course of Study*, the textbooks must have instructional content that all teachers can practice. At the same time, however, this has another implication, namely that the textbooks' content may lead to basic teaching but not to a conceivably ideal or optimal teaching. The textbooks should be understood to provide a minimum requirement of teaching.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

In this study we adopt the notions of didactic transposition and praxeology, which are main constructs of the Anthropological Theory of the Didactic (Chevallard, 2019). In ATD, every human activity is described as praxeology, and praxeology is categorized variously by the object of study. For example, mathematical praxeologies are related to a student's mathematical activities, and didactic praxeologies are related to a teacher's teaching activities. A praxeology ($\wp$) is constituted in both a practical block or praxis denoted by $\Pi=[T/\tau]$ and a theoretical block or logos denoted by $\Lambda=[\theta/\Theta]$: $\wp=\Pi\bigoplus\Lambda=[T/\tau]\bigoplus[\theta/\Theta]=[T/\tau/\theta/\Theta]$. The ordered pair of Π consists of type of tasks T, and technique τ for performing tasks $t\in T$. τ is related to each T, i.e., is the 'way of doing' T. The associated pair of Λ consists of technology θ which explains or justifies Π, that is, meaning *techne-logos* or 'the discourse on the τ', and theory Θ which elaborates the meaning of discourse to encompass the whole network of understandings and justifications used to account for the technology itself and its relation to other technologies (Rasmussen, 2016).

Kyozai-kenkyu is a teacher's activity for his/her classroom lesson(s) and is related to mathematical knowledge. Therefore, *kyozai-kenkyu* can be described in terms of both mathematical and didactic praxeologies, as well as being characterised as a *paradidactic praxeology,* which is about teachers' didactic activities *outside* the classroom (Miyakawa & Winsløw, 2013; Rasmussen, 2016). In terms of didactic transposition (Bosch & Gascón, 2006), mathematical knowledge involved in *kyozai-kenkyu* can be regarded as corresponding to different institutions such as scholarly knowledge, knowledge to be taught, taught knowledge, and learned, available knowledge. Here, what researchers can do is to present a reference epistemological model as a way of seeing such a transposition process in relation to mathematical knowledge involved in *kyozai-kenkyu*. Thus, the research questions in this study are (1) 'How can *kyozai-kenkyu* using textbooks be characterised as paradidactic praxeologies?' and (2) 'What aspects of *kyozai-kenkyu* can be linked with a didactic transposition process?'

A PRELIMINARY ANALYSIS OF TWO CASES

To answer our research questions, we briefly describe two different lessons (Case A and Case B), and then characterise different uses of textbooks for *kyozai-kenkyu* in terms of paradidactic praxeologies. In this study, however, we do not distinguish θ and Θ strictly in this paper, instead simply describe them as logos.

Case A is the lesson of adding fractions with the same denominator which has been implemented in the third grade (Yamamoto et al., 2015). Figure 1 shows a textbook page for this lesson. It involves formulating an equation from a word problem, explaining how to calculate using a number line model, and exercises. Figure 2 shows the blackboard writing in the classroom after the lesson. The main problem in this lesson is ‘There are four-tenths of a litre of coloured water and three-tenths of a litre of coloured water. How much coloured water is there in total?’ The blackboard context of this problem is similar to the textbook, but the teacher has slightly changed the numerical setting. He took students’ errors, like 4/10+3/10=7/20, into account, aiming at discussing them in the lesson. The reason for the numerical change was to make this discussion easier. In fact, at the beginning of the lesson, when students answered that the sum of 4/10+3/10 is 7/10, the teacher intentionally responded, ‘Isn’t it 7/20? Why not add denominators?’ This became the central topic in the next phase of the lesson. The students’ ideas and drawings in that phase were based on ‘the sum of four and three divided into ten’, which was the same as the mathematical activity intended by the textbook. While the textbook simply aims at thinking about *how* to add fractions, what the teacher intended was for the students to understand *why* such a way was right.

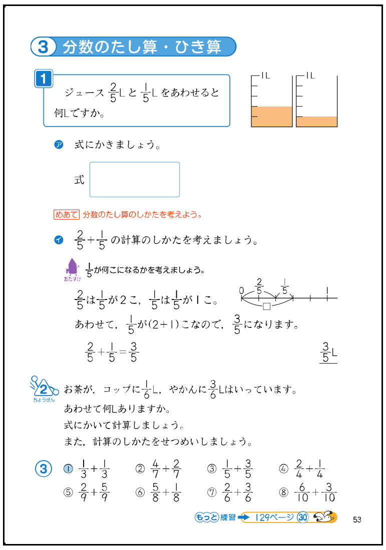


Figure 1. Textbook page, Case A (Shimizu et al., 2015, p. 53)

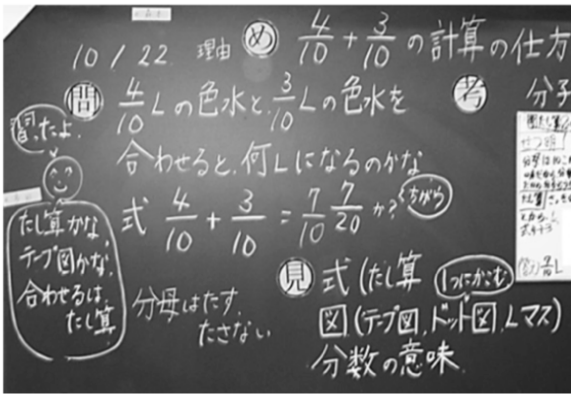
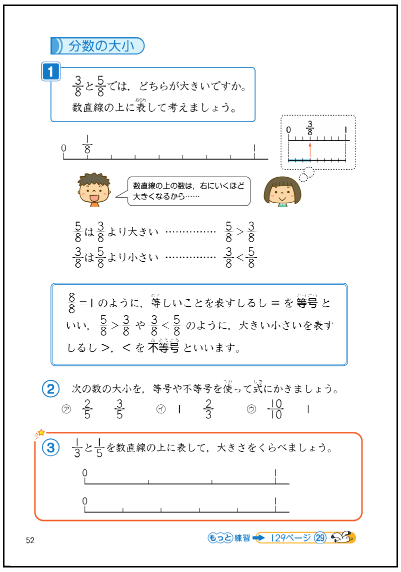


Figure 2. Blackboard in Case A

Case B is the lesson in comparison of fractions which has been implemented in the third grade as well (Maeda et al., 2016). Figure 3 shows the textbook page for this lesson. It involves investigating the magnitude of two fractions using a number line, introducing the symbol of inequality, and exercises. The main problem in the textbook is focused on ‘Compare two fractions (3/8 and 5/8)’. However, in the actual lesson, the teacher reconsidered the mathematical knowledge at stake. She set a new question, ‘Is there any other fraction between these two fractions?’ Because she placed significance on that, comparison of fractions according to the order structure of rational numbers appears somehow as the mathematical background. She also changed the numerical setting of the question (like 1/4 and 3/4) within the same context as the textbook. At the start of the lesson, students represented fractions by folding a tape and then could easily find a fraction between two fractions based on such actions (by further folding). At first, they found an ‘intermediate fraction’ with the tape, then gradually became able to process the problems without the tape (see Figure 4). In Case B, what the teacher aimed for in the lesson is a *new* mathematical goal beyond the textbook.



分数の大小

1 $\frac{3}{8}$と$\frac{5}{8}$では，どちらが大きいですか。
数直線の上に表して考えましょう。

0 $\frac{1}{8}$ 1

0 $\frac{3}{8}$ 1

数直線の上の数は，右にいくほど大きくなるから……

$\frac{5}{8}$は$\frac{3}{8}$より大きい ………… $\frac{5}{8} > \frac{3}{8}$
$\frac{3}{8}$は$\frac{5}{8}$より小さい ………… $\frac{3}{8} < \frac{5}{8}$

$\frac{8}{8} = 1$ のように，等しいことを表すしるし＝を等号といい，$\frac{5}{8} > \frac{3}{8}$ や $\frac{3}{8} < \frac{5}{8}$ のように，大きい小さいを表すしるし＞，＜を不等号といいます。

2 次の数の大小を，等号や不等号を使って式にかきましょう。
㋐ $\frac{2}{5}$ $\frac{3}{5}$ ㋑ 1 $\frac{2}{3}$ ㋒ $\frac{10}{10}$ 1

3 $\frac{1}{3}$と$\frac{1}{5}$を数直線の上に表して，大きさをくらべましょう。
0 1
0 1

52 もっと練習 ➡ 129ページ 29

Figure 3. Textbook page, Case B (Shimizu et al., 2015, p. 52)

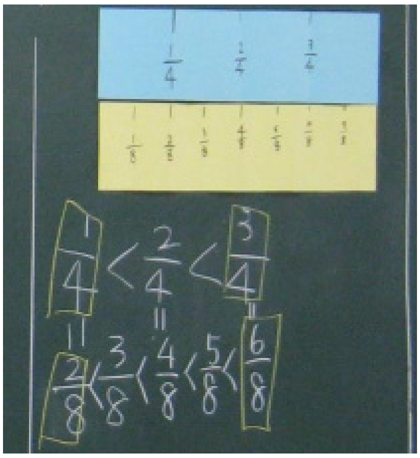


Figure 4. Blackboard in Case B

KYOZAI-KENKYU AS PARADIDACTIC PRAXEOLOGIES

In both cases A and B, teachers develop different lessons than the textbook pages with hoping for better learning of their students. However, it should be pointed out that the two approaches are different, especially in terms of textbook uses.

The teacher’s paradactic praxeology (PDP$_b$) in case A can be described as:

PDP$_b$: *Kyozai-kenkyu with textbook*

> T$_b$: Designing a lesson for students to understand why the way of adding fractions with the same denominator is right;
>
> τ_b: Changing the numerical values of the textbook problem for students to notice that adding denominators is inappropriate.

The logos that legitimates these T and τ is recognized as:

> θ_b/Θ_b: *Kyozai-kenkyu* is to get a better way of teaching based on students’ abilities and thinking; textbooks describe a standard way of teaching, which can be changed.

PDP$_b$ is commonly seen in the practice of lesson study. Many lesson studies aim at improving teaching and designing a task for the purpose of professional development. The focus is often directed to the teacher’s pedagogical skills, but substantial change in mathematical content is not necessarily demanded.

On the other hand, the phenomenon such as Case B typically seen in a developmental research practice (e.g., curriculum development), thus it does not occur as frequently in a community of teachers only. The teacher's paradidactic praxeology in Case B can be described as:

PDP_c: *Kyozai-kenkyu beyond textbook*

T_c: Designing a lesson for students to understand the substantial mathematical meaning of the order of rational numbers;

τ_c: Making it easy for students to know the density of rational numbers by formal calculation, starting with folding a tape;

θ_c/Θ_c: *Kyozai-kenkyu* is to get a deeper insight of mathematical knowledge; textbooks are not a set of absolute truths, but a source for designing an alternative content to be taught.

PDP_b and PDP_c are not necessarily ordinary works for Japanese teachers. Rather, their daily *kyozai-kenkyu* practice can be generally described as:

PDP_a: *Kyozai-kenkyu in textbook*

T_a: Designing a lesson for students to understand the content of textbooks;

τ_a: Treating the textbook task as carried in it for implementing a lesson well;

θ_a/Θ_a: *Kyozai-kenkyu* is to understand what the textbooks describe; textbooks have a certain authority and are always true.

DISCUSSION AND CONCLUSION

Generally, textbooks can be an empirical source for researchers to understand knowledge to be taught in the didactic transposition, and *kyozai-kenkyu* can be involved in the process from the knowledge to be taught to the taught knowledge. It is typically the case for PDP_b. However, our analysis shows there are some variations. For example, PDP_a does not involve such process because their taught knowledge is based on the textbook without any change. Therefore, PDP_a can be linked with the transposition process from the taught knowledge to the learnt knowledge. On the other hand, PDP_c can refer to the scholarly knowledge (e.g., a concept of density) and to its transposition process as shown in Figure 5.

The types of *kyozai-kenkyu* do not always appear based on teachers' professional skills or mathematical abilities; rather, the type depends on the community or the situation in which the teacher is actually involved. Therefore, we do not intend to say that one type is practiced by novice teachers and another by experienced

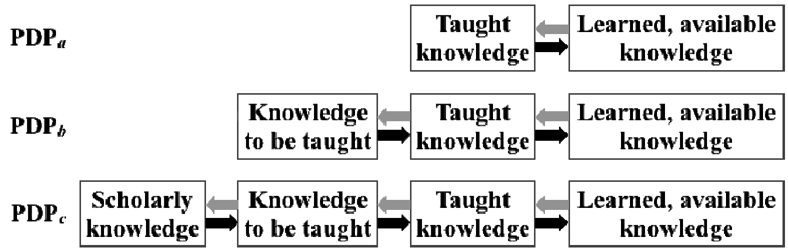


Figure 5. The PDPs and the didactic transposition processes

teachers. Since the analyses of the two lessons are very limited and the discussions rely on Japanese local contexts, more detailed analyses and further discussions based on international perspective are needed to justify the identified paradidactic praxeologies.

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References

Bosch, M., & Gascón, J. (2006). 25 years of the Didactic Transposition. *Bulletin of the International Commission on Mathematical Instruction*, *58*, 51–65.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114.

Fujii, T. (2015). The critical role of task design in lesson study. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study* (pp. 273–286). Cham: Springer.

Maeda, S., Yamanaka, N., & Mizoguchi, T. (2016). Improving teaching on comparing of fractions: focusing on the density of rational numbers. *Tottori Journal for Research in Mathematics Education*, *19*(5), 1–14. (in Japanese)

Miyakawa, T., & Winsløw, C. (2013). Developing mathematics teacher knowledge: The paradidactic infrastructure of "open lesson" in Japan. *Journal of Mathematics Teacher Education*, *16*(3), 185–209.

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: Didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education*, *19*(4), 301–324.

Shimizu, S., Funakoshi, S., Negami, S., & Teragaito, M. (Eds.) (2015). *Fun with math 3 for elementary school*. Osaka: Keirinkan. (in Japanese)

Shimizu, Y. (2014). Lesson study in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 358–360). Dordrecht: Springer.

Takahashi, A. (2016). Recent trends in Japanese mathematics textbooks for elementary grades: Supporting teachers to teach mathematics through problem solving. *Universal Journal of Educational Research*, *4*(2), 313–319.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective Lesson Study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *AMTE Monograph 5: Inquiry into mathematics teacher education* (pp. 131–142). Information Age.

Yamamoto, M., Togasaki, S. & Shinno, Y. (2015). On the teacher's didactic acts of encouraging social interactions during the lesson: through observing a lesson of "Fractions" in the third grade. *Osaka Journal of mathematics education*, *44*, 27–34. (in Japanese)

THE VALUE OF TECHNOLOGY IN CHILEAN SCHOOL MATHEMATICS TEXTBOOKS: A WAY OF CONDUCTING CONDUCTS

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This paper aims at exploring the value given to technology in the Chilean school mathematics textbooks. In recent years, digital media has been incorporated as an important part of the school mathematics textbook, contributing to the segregation of students and to widening the gap that exists among students coming from different socioeconomic backgrounds. This phenomenon has unfolded forms of producing a particular reality, as well as, conducting students' and teachers' ways of thinking and acting. That is, the use of technology has an effect on how students and teachers experience and perceive school mathematics. Within the materials provided by the Chilean Ministry of Education, it is possible to identify a gap between the use and representation of technology—its value—and the technology—actual technological devices—needed for the proper implementation of the activities suggested in these materials.

INTRODUCTION

This paper aims at problematizing, from a socio-political perspective, how it is producing a particular reality, as well as, governing students' and teachers' ways of thinking and acting. The unfolding of this paper sets an exploratory approach by following a rhizomatic (Deleuze & Guattari, 1987) analytical strategy. The rhizome enables to unfold a network of multiple dimensions that allows considering materials from the role of the media in the mediatisation of education as well as research that has argued on the market as part of the neoliberal structuring of education. In this light, the data gathered for this paper, taken as the first approximation to the research problem, corresponds to media reports that have addressed the use and incorporation of the technology and the Chilean school mathematics textbooks, in order to explore the value given to technology in textbooks—the ways of naturalizing its use and importance—and the segregation shaped by that textbooks' use of technology.

It is an exploratory approach since the "economic spectrum" of education is not a usual object of study within research in the field of mathematics education. Considering the released of ICME 13th monograph "Research on mathematics textbooks and teachers' resources: Advances and issues" (Fan, Trouche, Qi, Rezat & Visnovska, 2018), the socio-political perspective on mathematics education has not been widely explored despite *sociopolitical turn* of mathematics education (Gutierrez, 2013).

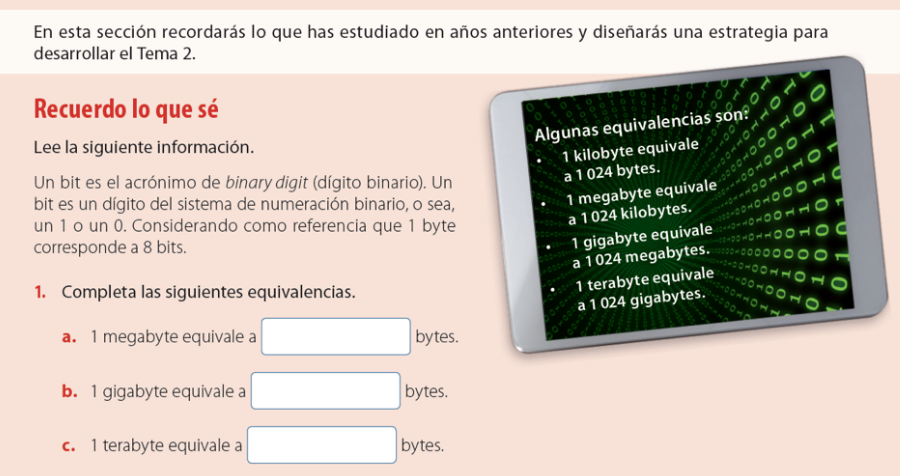
In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 263–268). Paderborn: Universitätsbibliothek Paderborn.

The interest of looking at school mathematics textbooks, from this perspective, comes from the Chilean struggles to fine-tune educational policies towards providing equal opportunities to all students. And also, because school textbooks are more and more incorporating complementary digital resources and promoting the use of the new technology in the school. We propose that the use and incorporation of the technology in the Chilean school mathematics textbooks must be understood as a way of governing, where students' and teachers' subjectivities are drawing, by conducting their conduct—by delineating and shaping their ways of being and acting.

Currently, within diverse social and academic spheres it has been argued the importance of the use and incorporation of technology in the teaching and learning of mathematics, as well as in the new possibilities that this opens (see the works published in Hoyles & Lagrange, 2010; MINEDUC, 2016)—Bornand (2013) asserts that the digital textbooks characterize by promoting the self-regulation of learning, problem-solving, motivation and reflection. Even more, technology has taken a special impact and importance—value—when talking about innovation and improvement of teaching and learning practices (Eyyam & Yaratan, 2014; MINEDUC, 2016). For example, in the Chilean context, The Ministry of Education of Chile (MINEDUC) in partnership with Discovery Education implement a pilot project about digital school textbooks (see https://www.mineduc.cl/201810/01/texto-escolar-digital-proyecto-piloto/), the goal of this project is to improve the quality of learning. MINEDUC asserts that the use of technology is key in the teaching of abilities and knowledge—such as mathematical abilities and knowledge—that require the citizen of the twentieth century. Moreover, MINEDUC has implemented an initiative called *I connect to learn* (Me conecto para aprender), which has the aim to close the gap of accessing and using of information and communication technologies, and in this way, to support the learning process (see http://meconecto.mineduc.cl/).

THECNOLOGY AND CHILEAN SCHOOL MATHEMATICS TEXTBOOKS

Navigating by Chilean school mathematics textbooks, it is possible to identify three nuances in the use and incorporation of the technology. Firstly, technology is utilized as a *visual recourse*, there are diverse representations of devices technological in order to exemplify or to contextualize some activities, notion or ideas. For example, the use of imagen of computers (e.g., Galasso, Maldonado & Marambio, 2016, p. 10), calculators (e.g., Galasso, Maldonado & Marambio, 2016, p. 67) and smartphones (e.g., Chacón, García, Rupin, Setz, & Villena, 2018, p. 76), among others. These representations are characterized by showing the wonders of technology (See Figure 1).


En esta sección recordarás lo que has estudiado en años anteriores y diseñarás una estrategia para desarrollar el Tema 2.

Recuerdo lo que sé

Lee la siguiente información.

Un bit es el acrónimo de *binary digit* (dígito binario). Un bit es un dígito del sistema de numeración binario, o sea, un 1 o un 0. Considerando como referencia que 1 byte corresponde a 8 bits.

1. Completa las siguientes equivalencias.

a. 1 megabyte equivale a ______ bytes.

b. 1 gigabyte equivale a ______ bytes.

c. 1 terabyte equivale a ______ bytes.

In this activity is soliciting to calculate the equivalence between megabyte and bytes, gigabyte and bytes, and terabyte and bytes. Here, the tablet is used as an aide-memoire, showing some equivalences

Figure 1. Use of tablet as an aide-memoire

Secondly, technology is utilized as an *exploratory recourse*, textbooks invite students to *"use the calculator for approximating"* (e.g., Chacón, et al., 2018, p. 24) and to *"visit the web..."*, for example. Here, it is looking for that students have a first approximation to some concepts or procedures. And thirdly, as a *complementary resource*, throughout diverse unities, that are developed on textbooks, are promoted, on the one hand, that students and teachers make use of *"complementary digital resources"* (e.g., Chacón, et al., 2018, p. 27; Galasso, Maldonado, & Marambio, 2016, p. 31) designed for each unity—this resource is composed of didactic material and software that are given to teachers through a USB memory. On the other hands, the textbooks promote that students *"visit the web..."* (Galasso, Maldonado, & Marambio, 2016, p. 79) for deepening some contents or ideas.

Moreover, in the Chilean context, within the mathematics teacher's assessment, the use of technology is assessed, as well as the use of the textbook (MINEDUC, 2016). Currently, it would seem that a good teacher must have the tools, knowledge and competencies for using the technology in his/her practices, as well as, promoting the development of certain desired tools and abilities that the citizen of the twentieth century must have—in his/her students, the management and the use of technology.

THE GAP BETWEEN THE USE AND REPRESENTATION OF TECHNOLOGY AND THE IMPLEMENTATION OF ACTIVITIES

In the Chilean context, most of the schools in Chile have a high number of students considered vulnerable (see www.junaeb.cl/ive?lang=en). Many of those students do not have access to technology or to last technology. 99% of whom conform the more affluent Chilean social group has smartphones and 99% of them have internet access in their homes. In contrast to whom conform the less affluent Chilean social group, 43% of them have smartphones and 52% of them have internet access in their homes (El Mercurio, 2019). Even more, a large number of schools do not have access to a good level of technology. In some cases, some schools do not have the minimal conditions for applying technology in the classroom, or for implementing what the textbook suggests as *"complementary digital resources"*. In other words, in the Chilean context, it is not complicated to find schools without access to computers or internet connection in the classroom—amongst other things—or students without access to technology in their house—for example, internet connection. In this context, how is it possible to implement the activities suggested in the textbooks? And how is it possible to ensure an education for all, if there are certain socio-economic conditions that make it impossible?

The textbooks and their applicability are thinking from standardized ideas, an ideal reality, and desired subjects. But it is naïve to think that all schools' conditions are the same—which seems that is a principle of standardized education—and that all students and teachers have the same opportunities and access to technology. Here is where we ask: what happens when the use and incorporation of technology in the textbooks become a problem? Problem because it is promoting a gap between who has or does not have access to technology, as well as promoting a reality that is far from students and teachers or that does not represent them. In other words, it is widening the gap that exists among students coming from different socioeconomic backgrounds. The implementing of or not implementing the activities promoted by textbooks could be related to increasing the difference in students' performance and outcomes. It has shown and studied that students with better socioeconomic levels—thus, with access to better didactic materials—have better output in standardized tests. The socioeconomic background and access and use of technology are affecting how students and teachers experience and perceive school mathematics.

Also, regarding the implementation of the activities promoted by textbooks, these conduct to the reality of schools, students and teachers. Consequently, the implementation of activities is in function of who has or have not access to what is desired. Wherein the desired—normal or standard—is constantly confronted against the real school, student and teacher. Finally, the use and incorporation of technology in the textbooks shapes a kind of utopia—*technotopia*. Due to the use of technology is producing and reproducing a reality that is not for all, only for who can access to technology, but it is a reality desired by society. In this reality, the technology helps to approach to what is desired by society.

A WAY OF CONDUCTING THE CONDUCT

The incorporation and the use of technology in the textbooks have played a key role in the constitution of conditions, in which the students' and mathematics teachers' conduct are conducted, by delineating and shaping their ways of thinking and acting, as well as the segregation of students. By shaping standards that the desired subject—student and teacher—must achieve, around of certain tools, abilities, and knowledge needed for current society—a society characterized by being in constant change. Currently, the subject that can incorporate and use technology in his/her life has a "value-added" for society. Students and teachers and their becoming are governed—in terms of Foucault—through the illusion drawn the *technotopia*. Illusion based on the idea that technology favours the constitution of a better, more just, egalitarian, and developed society. It asserts that with technology is possible to close the gaps if all are considered equal and with the same possibilities. Technology performs a gate-keeping role.

The power of technology in the textbooks lies in the capacity of characterizing the desired student and teacher, as well as the desired reality. This characterization of desired expresses a double gesture of hope and fear. The hope that students and teachers embody ways of thinking and acting, framed in dominant rationality. But, at the same time, the fear that they will not be achieved. The becoming of students and teacher is the key, since, "[t]he actual is not what we are but, rather, what we become, what we are in the process of becoming —that is to say, the Other, our becoming-other." (Deleuze & Guattari, 1994, p. 112).

References

Bornand, S. (2013). Avances, experiencias y futuro de los textos escolares digitales. In *Nuevas Ideas en Informática Educativa TISE 2013*. 30–35

Chacón, A., García, G., Rupin, P., Setz, J., & Villena, M. (2018). *Matemáticas 2° Medio: texto del estudiante*. Santiago, Chile: Ministerio de Educación-SM

Deleuze, G., & Guattari, F. (1987). *A thousand plateaus: capitalism and schizophrenia*. Minneapolis: University of Minnesota Press.

Deleuze, G., & Guattari, F. (1994). *What is philosophy?* New York: Columbia University Press.

El Mercurio. (2019). *El perfil de los siete grupos socioeconómicos de la nueva segmentación y como se divide la población de Chile.* Retrieved from https://www.emol.com/noticias/Economia/2018/10/19/924437/El-perfil-de-los-siete-grupos-socioeconomi cos-de-la-nueva-segmentacion-y-como-se-divide-la-poblacion-de-Chile.html

Eyyam, E., & Yaratan, H. (2014). Impact of Use of Technology in Mathematics Lessons on Student Achievement and Attitudes. *Journal of Social Behavior and Personality*, *42*, 31–42.

Fan, L., Trouche, L., Qi, C., Rezat, S., & Visnovska, J. (Eds.). (2018). *Research on mathematics textbooks and teachers' resources: Advances and issues*. Cham: Springer.

Galasso, B., Maldonado, L., & Marambio, V. (2016). *Matemáticas 1° Medio: texto del estudiante*. Santiago, Chile: Ministerio de Educación-Santillana.

Gutierrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Hoyles, C., & Lagrange, J. B. (2010). *Mathematics education and technology: Rethinking the terrain. The 17th ICMI Study*. Berlin: Springer.

Miñana, C., & Rodríguez, J. (2003). La educación en el contexto neoliberal. In D. Restrepo (Ed.) *La falacia neoliberal. Críticas y alternativas*. Bogotá, Universidad Nacional de Colombia.

MINEDUC (2016). *Docentes en chile: conocimiento y uso de las tic 2014*. Santigo, Chile: Centro de Estudios-MINEDUC.

Pais, A. (2012). A critical approach to equity. In O. Skovsmose, & B. Greer (Eds.) *Opening the cage. Critique and politics of mathematics education* (pp. 49–91). Rotterdam: Sense.

THE CONVERSION ACTIVITIES BETWEEN REPRESENTATION REGISTERS AS AN INSTRUMENT FOR THE ANALYSIS OF MATHEMATICS TEXTBOOKS

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Semiotics raises an increasing interest in Mathematical Education since Semiotics provides an array of methods and concepts to understand the nature of mathematical discourse. Signs are carriers of cultural conventions and Semiotics helps to understand the way in which individuals think and communicate with signs in a cultural context. Duval's theory of semiotic representations characterizes Mathematical learning in terms of student ability to handle several representation registers of mathematical concepts. Using as an example a concept of higher mathematics, the gradient, in this work we suggest as criterion for the analysis of textbooks to analyze the existence of activities that require making conversions between representations of the concept among the diverse representation registers.

INTRODUCTION

Research on textbooks is becoming an efficient method for the study of teaching-learning processes. This research has led to a wide variety of criteria for analyzing textbooks such as the search for possible epistemological obstacles in the introduction of a certain concept or the characterization of types of mathematical tasks that are proposed (Sidenvall, Lithner, & Jäder, 2015; Wijaya, Van den Heuvel-Panhuizen, & Doorman, 2015; Gordillo, Pinzón, & Martínez, 2017; Díaz-Levicoy & Arteaga, 2017; Gea, López-Martín, & Roa, 2015).

There is also extensive research concluding that the ability of students to work in coordination with multiple representations of a concept is a critical skill and sign of excellence in mathematics (Duval, 2006, De Bock, Van Dooren, & Verschaffel, 2015). Consequently, a plausible approach to the qualitative analysis of a textbook is to determine if it includes conversion tasks among the various representations (registers) considered usual to work with a certain concept. This perspective requires further research because, as indicated by Chang, Cromley, and Tran (2016), little is known about the prevalence of coordination tasks in such textbooks. In this regard, Hughes-Hallett et al. (2017, p. ix, 2) propose the "Rule of Four", in which notions are presented graphically, numerically, symbolically and verbally. Chang, Cromley, and Tran (2016) also use fundamentally such record classification. However, we think that graphic register denomination is too general and hence, for a given concept, it might be necessary to name more specific graphic registers. The qualitative research we present is a contribution in this direction and answers the following questions:

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 269–274). Paderborn: Universitätsbibliothek Paderborn.

1. Is it possible to characterize the semiotic registers that are used to work with the concept of gradient?

2. In the textbook by Larson, Hostetler, and Edwards (2002), are proposed conversion activities between such semiotic registers?

THEORETICAL FRAMEWORK

Semiotic representation registers

According to Duval (2006), a system of semiotic representations or registers is an arrangement of signs and associations (including the language) that are generated according to specific rules. The registers facilitate understanding and enable to describe, communicate and produce new knowledge through mathematical processing.

Treatment and conversions

Two types of semiotic representation transformations of objects regulate thinking processes in Mathematics: treatments and conversions. The treatments are transformations that take place within a unique register when applying the rules valid in that register to a representation of the object. On the contrary, conversions are transformations that consist of switching from a representation of the object represented in a certain register (source register) to another representation of the same object represented in a different register (destination register). See a more detailed description of these concepts in Duval (2006). For example, Figure 1 shows three semiotic systems to represent and operate with non-integer numbers: graphical, decimal and fractionary. Note that these semiotic representations not only allow expressing or communicating the notion of fraction, but also allow generating new knowledge. For example, obtaining treatments to operate within the various number representations.

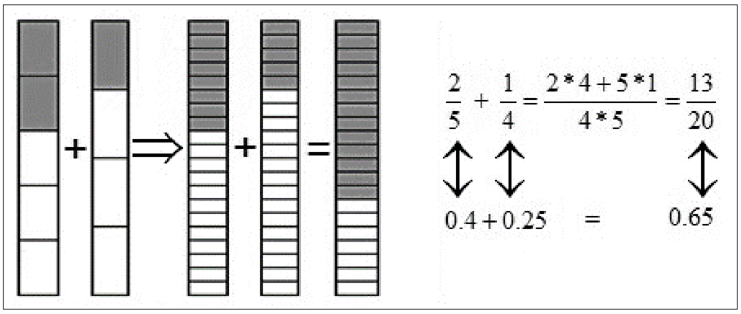


Figure 1. Three representation registers for non-integer numbers and examples of treatments (operations + and *) and conversions (↔) within those registers.

That change of registers implies a challenging action for students. Indeed, Duval points to such transformation as the root of the mathematics learning problem. In this regard, as indicated by Rico (2009) and D'Amore, Pinilla, Iori, and Matteuzzi (2015), the combined use (in both directions) of multiple representation systems is required in order to fully understand each concept or mathematical structure.

METHODOLOGY

Representation register for the gradient of $f(x, y)$ concept

International reference Calculus textbooks (Larson, Hostetler, & Edwards, 2002; Smith & Minton, 2000; Marsden & Tromba, 2004; Thomas, Finney, Weir, & Giordano, 2003; Stewart, 2005; Hughes-Hallett et al., 2017) have been studied and the following five representation registers that implicitly appear in all of them have been characterized.

Algebraic-verbal register (AV)

Register in which natural and formal languages are used. In this register algebraic expressions are formulated and processed, problems are set out, concepts are formally defined, and theorems constituting the properties that have such concepts and the relationships existing between them are demonstrated.

Level curves register (LC)

It is defined by a family of curves corresponding to the equation $f(x, y) = C$, usually with a constant step between two consecutive curves.

Gradient map register (GM)

If $f(x, y)$ admits partial derivatives, the gradient vector in P, $\nabla f\,(P) = (fx(P), fy(P))$, is represented as an oriented arrow showing its direction, sense and magnitude.

Maximum increment curves (MI)

The C curve that passes through a given $P(a, b)$ point and which tangent vector is parallel to the gradient vector of $f(x, y)$ is represented. Note that the AV→MI conversion requires the resolution of the following differential equation system: $x'(t) = f_x(x, y);\ y'(t) = f_y(x, y); x(t_0) = a;\ y(t_0) = b.$

3D surface register (3D)

The surface S defined by $z = f(x, y)$ is represented by means of the traces of S in parallel planes to the coordinate planes.

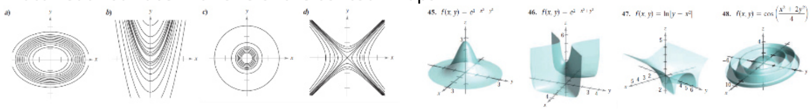
Sample

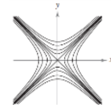
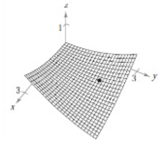
The textbook by Larson, Hostetler, and Edwards (2002) has been selected. In this work, we consider the conversions between the algebraic-verbal register and the rest of the registers.

RESULTS AND CONCLUSIONS

For the selected textbook, the following table gathers the activities that work the conversions indicated in section 2.1. For each activity, the working conversion is defined, as well as the corresponding location within the text and a brief description and an example of the proposed activity.

Table 1. Selected activities from Larson, Hostetler, and Edwards (2002).

Conversion	Location	Description of the proposed activity	Example
AV → LC	Ex. 49-56, 77, 81b Section 12.1 Pages 199-201	Sketch some level curves for given $f(x,y)$ functions.	The temperature T (in degrees Celsius) at any point (x,y) of a circular plate of 10 meters radius is $T = 600 - 0{,}75x^2 - 0{,}75x^2$, where x, y are measured in meters. Draw some isothermal curves.
	Ex. 45-48 Section 12.1 Page199	Correlate some contour maps with their corresponding $f(x,y)$ functions.	Match each surface with one of the contour maps 
AV → GM	Ex. 46, 50 Section 12.6 Pages 245-246	Discuss the geometric meaning of the gradient vector and an orthogonal direction to it.	Find a unit vector u orthogonal to $\nabla f(1,2)$ and calculate $D_u f(1,2)$. Discuss the geometrical meaning of the result. $f(x,y) = 9 - x^2 - y^2$
	Ex. 54b, 59-62 Section 12.6 Page 246	Sketch the gradient vector after having had calculated it analytically.	The figure shows a level curve of the function $f(x,y) = \frac{8y}{1+x^2+y^2}$. Sketch, for the $(\sqrt{3},2)$ point of the level curve, the vector in the direction of maximum rate of increase of the function.
AV → MI	Ex. 73-74 Section 12.6 Page 247	Given a $f(x,y)$ function, calculate the maximum increment path.	Find the path followed by a thermal tracker located at point P of a metal plate which temperature field is $T(x,y)$: $T(x,y) = 400 - 2x^2 - y^2$ $P(10,10)$
AV → 3D	Ex. 31-38, 43a Section 12.1 Pages 198-199	Sketch the surface corresponding to $f(x,y)$ functions.	Draw the surface given by the following function: $z = 4 - x^2 - y^2$
	Ex. 39, 47 Section 12.6 Pages 245-246	Sketch the surface corresponding to $f(x,y)$ functions.	Draw the graph of f in the first octant and mark the point $(3,2,1)$ over the surface. $f(x,y) = 3 - \frac{x}{3} - \frac{y}{2}$

	Ex. 45-48 Section 12.1 Page 199	Associate contour maps and their corresponding $f(x,y)$ functions.	*See the second example of conversion AV →LC.*
LC→ AV	Ex. 83 Section 12.1 Page 201	Given a contour diagram, interpret which are the points of maximum and minimum height value, as well as the point of maximum speed of change.	The meteorologist measure the atmospheric pressure in millibars. From their observations, they draw up weather maps indicating constant atmospheric pressure curves (isobars). Knowing that the closer the contour lines are together the greater the wind speed, associate to points A, B and C (a) the maximum pressure, (b) the minimum pressure and (c) the maximum wind speed. 
	Ex. 69, 70 Chapter 12 Page 283	Describe the most relevant features of the $f(x,y)$ function represented by a given contour diagram.	Write a short paragraph about the surface whose level curves (with values of c uniformly spaced) are given in the figure. Comment [...] the magnitude of the gradient, etc. 
3D → AV	Ex. 51a, 52a Section 12.6 Page 246	Given a $3D$ graph of a $f(x,y)$ function, estimate the components of a vector parallel to the gradient.	Use the graph to estimate the components of the vector in the direction of maximum rate of increase of the function at the indicated point. 

As can be seen, the analyzed textbook proposes conversion activities between AV and most other registries. The only weakness would be for MG → AV and MI → AV conversions.

In mathematics education the ability of students to coordinate the multiple representation registers of a mathematical concept is considered of relevance. Therefore, it is of interest to analyze whether textbooks include coordination tasks between registers. However, this analysis methodology is still developing. In this sense, this work is a contribution to this methodology. In the first place, we suggest that it is necessary to characterize the semiotic registers that are used to work with a certain concept. Secondly, it has been demonstrated how the register characterization allows to accurately describe the conversion tasks proposed in the textbook. This work has been carried out for the gradient concept and has been applied to a textbook, even though only partially (conversions between AV and the other registers). In the next phases of the project, it will be necessary to study whether conversion tasks are proposed among the remaining pairs of registers and to extend the study to other textbooks.

References

Díaz-Levicoy, D., & Arteaga, P. (2017). Conflictos semióticos potenciales sobre gráficos estadísticos en libros de texto chilenos de Educación Primaria. In J. M. Contreras, P. Arteaga, G. R. Cañadas, M. M. Gea, B. Giacomone, & M. M. López-Martín (Eds.), *Actas del Segundo Congreso International Virtual sobre el Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemáticos.* Retrieved from enfoqueontosemiotico.ugr.es/civeos.html

Chang, B. L., Cromley, J. G., & Tran, N. (2016). Coordinating multiple representations in a reform calculus textbook. *International Journal of Science and Mathematics Education*, *14*(8), 1475–1497.

De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: Two studies on the role of external representations. *International Journal of Science and Mathematics Education*, *13*(1), 47–69.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131.

Gea, M., López-Martín, M., & Roa, R. (2015). Conflictos semióticos sobre la correlación y regresión en los libros de texto de Bachillerato. *Avances de Investigación en Educación Matemática*, (8).

Gordillo, W., Pinzón, W. J., & Martínez, J. H. (2017). Los mapas conceptuales: Una técnica para el análisis de la noción de derivada en un libro de texto. *Formación Universitaria*, *10*(2), 57–66.

Hughes-Hallett, D., Gleason, A. M., Lock, P. F., Flath, D. E., Gordon, S. P., Lomen, D. O., ... & Tecosky-Feldman, J. (2017). *Applied calculus*. Hoboken, NJ: John Wiley & Sons.

Larson, R., Hostetler, R. P., & Edwards, B. H. (2002). Cálculo (7ª edición). *Madrid, España:* Ediciones Pirámide

Marsden, J. E., & Tromba, A. J. (2004). *Cálculo vectorial* (5º Edición). Pearson Educación.

Sidenvall, J., Lithner, J., & Jäder, J. (2015). Students' reasoning in mathematics textbook task-solving. *International Journal of Mathematical Education in Science and Technology*, *46*(4), 533–552.

Smith, R., & Minton, R. (2000). *Cálculo*. McGrawHill

Stewart, J. (2008). *Cálculo de una variable. Transcendentes y tempranas,* 6ª Edición, México D.F.: Cengaje Learning.

Stewart, J. (2012). *Cálculo de varias variables, trascendentes tempranas*. México D.F.: Cengage Learning.

Thomas, G. B., Finney, R. L., Weir, M. D., & Giordano, F. R. (2003). *Thomas' calculus*. Reading: Addison-Wesley.

Wijaya, A., Van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41–65.

THE FUNCTIONS OF A HIERARCHICAL CLASSIFICATION OF QUADRILATERALS IN JAPANESE TEXTBOOK: ITS PRESENTATION AND LIMITATION

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The purpose of this paper is to clarify functions of a hierarchical classification of quadrilaterals in Japanese textbooks. From the perspective of the five functions of a hierarchical classification of quadrilaterals (de Villiers, 1994), geometry in the latest Japanese lower secondary school textbooks was analysed. While the three functions related to problem solving were identified, the other two functions related to exclusive and alternative definitions were not identified in the textbooks. The results of analysis imply that there is a limitation that contradictory definitions must not be presented in a textbook, and that textbooks can present only a part of defining activities.

INTRODUCTION

In mathematics teaching and learning, we expect that students understand not only mathematical contents but also *functions* of the contents. Functions of different contents are pointed out in mathematics educational research (e.g., de Villiers, 1990; de Villiers, 1994). It is worth examining how the functions of contents are reflected in school mathematics textbooks. For example, Miyakawa (2017) explored the functions of proof in French and Japanese lower secondary school textbooks. In this paper, we focus on a hierarchical classification. Although a hierarchical classification is mathematically important, there is a fact that many students don't prefer it (de Villiers, 1994). It seems that the textbook writers tackle to this problem. In this paper, we would like to clarify how the textbook writers try to reflect functions of a hierarchical classification in the textbooks.

THEORETICAL FRAMEWORK

A hierarchical classification is particularly related to mathematical definitions. Freudenthal (1971) pointed out teaching mathematical defining activities rather than mathematical definitions themselves, as follow.

> Definitions are not preconceived to derive something from them, but more often they are just the last element of the analysis, the finishing touch of organizing a subject. Children should be granted the same opportunities as the grown-up mathematician claim for himself. Telling a kid a secret he can find out himself in not only bad teaching, it is a crime. (Freudenthal, 1971, p. 424)

From this standpoint, de Villiers (1994) proposed the five functions of a hierarchical classification of quadrilaterals as follows. First, a hierarchical classification leads to more economical definitions of concept and formulations of theorems. For example, if

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 275–280). Paderborn: Universitätsbibliothek Paderborn.

we choose an inclusive definition of a trapezoid, we can say that it is a quadrilateral with an axis of symmetry through at least one pair of opposite sides. On the other hand, if an exclusive definition, we need to add the condition that it has not a right angle. Second, a hierarchical classification simplifies the deductive structure and derivation of the properties of more special concepts. For example, if we classify rhombi as kites, all the theorems which have already been proved for kites are immediately made applicable to the rhombi. On the other hand, if not so, we need to prove the theorem related to rhombi again. Third, a hierarchical classification often provides a useful conceptual schema during problem solving. For example, if we would like to prove that a kite with one pair of opposite sides parallel is a rhombus, it is sufficient to prove that it is a parallelogram, because the rhombi are the intersection between the kites and the parallelograms. Forth, a hierarchical classification sometimes suggests alternative definitions and new propositions. For example, by using a hierarchical perspective, we can also define an isosceles trapezoid as a trapezoid inscribed in a circle, or as a trapezoid with at least one pair of opposite sides parallel. Fifth, a hierarchical classification provides a useful global perspective. For example, since the rhombi are the intersection between the kites and the parallelograms, from the diagonal properties of kites and parallelograms, we can get the property that the diagonals of a rhombus perpendicularly bisect each other.

Based on these functions, de Villiers (1998) pointed out that students can understand the advantages of a hierarchical classification of quadrilaterals through defining activities. According to these studies, it is critically important for students to engage in mathematical defining activities. There is, however, a dilemma that even if we would like to emphasize defining activities, we must present ready-made definitions in textbooks. It is worth examining how the functions of a hierarchical classification are reflected in school mathematics textbooks by the textbook writers.

Therefore, we can give the research question: how the five functions of a hierarchical classification of quadrilaterals are presented in Japanese school mathematics textbook?

METHOD

In Japanese educational system, private companies publish all textbooks, and MEXT check and approve them. In 2019, there are seven companies publish lower secondary school mathematics textbooks (from seventh grade to ninth grade). In case of public lower secondary school, Boards of Education in the districts adopt the textbooks.

In this paper, we analyse lower secondary school mathematics textbooks of two publisher: *Tokyo Shoseki* (Fujii et al., 2016) and *Keirinkan* (Okamoto et al., 2016). In lower secondary school, the current share of *Tokyo Shoseki* is 33.0% and the one of *Keirinkan* is 38.5% (Jiji Press, 2015). The reason why we analyse the textbooks of only two publishers is that MEXT determines curriculum standards and authorize all textbooks. It seems to be sufficient to analyse textbooks of these publishers for clarifying the characteristics of Japanese mathematics textbooks.

In the current Japanese mathematics curriculum, special quadrilaterals such as a square and a rectangle are placed to each grade as follows. Second grade students start learning concepts of triangles and quadrilaterals. After that, they learn concepts of a right angle through folding paper. In this way, they learn concepts of a square and a rectangle. Fourth grade students learn concepts of perpendicular and parallel of lines through using a set-square. By relating to parallel lines, they learn concepts of a trapezoid, a parallelogram, and a rhombus through constructing and tessellating. After that, they learn concepts of area especially formula for the area of a square and a rectangle. Fifth grade students engage in problems how to find the area of a parallelogram, a triangle, a trapezoid, and a rhombus. Eighth grade students learns mathematical proofs, definitions, and theorems. They learns mathematical definitions of a parallelogram, a rhombus, a rectangle, and a square in learning proof. After this learning, students apply the definitions and properties of these special quadrilaterals to different things.

The target descriptions in the textbooks are definitions and theorems of a quadrilateral, a parallelogram, a rhombus, a rectangle, and a square (not restrict within strictly mathematical definitions and theorems), and problems related to quadrilaterals (include example problems).The steps of analysis are as follows: (1) extracting the target descriptions from the textbooks, (2) collating between the descriptions in the textbooks with the explanations and the examples in de Villiers (1994), (3) determining whether the functions are identified or not in the textbooks.

RESULTS

The results of analysis are showed in Table 1. These are the results of only *Tokyo Shoseki*, because the ones of *Keirinkan* are not essentially different from ones of *Tokyo Shoseki*. The results consist of three parts: (a) the function was identified in eighth grade textbooks, (b) the functions were identified in ninth grade textbooks, and (c) the functions were not identified in any textbooks.

Firstly, we identified the functions of a useful conceptual schema during problem solving and a useful global perspective in eight grade textbook, which were related to problem solving for special quadrilaterals. Eighth grade students learn formal proofs in geometry. These functions were not presented to students directly, but embedded in the situations of geometrical proof problem. In this way, students are expected to acquire useful conceptual schema and global perspective of inclusive relations between quadrilaterals.

Secondary, we identified the function of simplification of deductive structure in ninth grade textbook, which was related to proving a theorem that if the midpoints of the four sides of a quadrilateral are consecutively connected, then the connected quadrilateral is a parallelogram. In the textbook, there is a dialogue between two students Sakura and Yuto. Sakura thinks that the connected quadrilateral is a parallelogram even if any quadrilateral is. Yuto wonders whether it is correct or not when the connected quadrilateral becomes a rhombus. Similar to the previous case, these functions were not presented to students directly, but embedded in the situations of geometrical proof problem. In this way, students are expected to acquire simplification of deductive structure related to special quadrilaterals.

Lastly, we did not identify the functions of economical definitions and formulations of theorems, and alternative definitions and new propositions in the textbooks. For example, it is necessary for us to compare to exclusive definitions in order to recognize the advantages of inclusive definitions. It is also necessary to select a statement as a definition in order to understand the characteristics of alternative definitions. However, the Japanese textbooks didn't explicitly deal with exclusive definitions or equivalent definitions of quadrilaterals. Therefore, we did not identify it in the textbooks.

DISCUSSION

From the results of analysis, we can classify two types of functions of a hierarchical classification of quadrilaterals in textbooks. The first type is functions which are related to situations of solving proof problems. As Japanese textbooks value and emphasize a problem solving situation in geometrical proofs (Fujita & Jones, 2014), this type can be presented in textbooks. Therefore, Japanese textbooks try to teach not only a hierarchical classification of quadrilaterals but also the functions of it through problem solving situations.

The second type is functions which have difficulty presented in textbooks. As the above, it is critically necessary for us to deal not only with one definition but also with more than two definitions to recognize the advantages of inclusive definitions and to understand the characteristics of alternative definitions. However, the Japanese textbooks didn’t explicitly deal with exclusive definitions or equivalent definitions of quadrilaterals. This is because we cannot give *contradictory two definitions* to the same concept *in a textbook* (In case of an extension of a concept, we can give two definitions to the same concept in a textbook). For example, if we give an inclusive definition and an exclusive definitions to the same concept in a textbook, these definitions are contradict. Therefore, there is a limitation that the *only non-contradictory definitions* to the same concept must be presented in a textbook. Although choosing one definition from equivalent definitions of a concept is an important defining activity, textbooks can present only a part of defining activities.

As Freudenthal (1971) said, we should distinguish mathematical definition as a product and mathematical defining as a process. In defining processes, we examine a lot of properties of the objects and construct several equivalent statements, after that we select a statement as a definition. In case of quadrilaterals, we can focus on different components of a quadrilateral such as sides, angles, diagonals, parallel lines, axes of symmetry. There is a fact that we can acquire different classifications and systems of quadrilaterals (Usiskin & Griffin, 2008). Based on the relativity of mathematical definitions, we should give students opportunities to define quadrilaterals by themselves (Zandieh & Rasmussen, 2010; Kobiela & Lehrer, 2015).

The functions	The descriptions in the textbooks (Translated by the author)	Identified functions
(i) Economical definitions and formulations of theorems	[Not identified] The fact that inclusive definitions or theorems about them are relatively simple than exclusive definitions is not presented, because the Japanese textbooks don't deal with exclusive definitions of quadrilaterals explicitly.	
(ii) Simplification of deductive structure	[Ninth grade (Problem)] Draw a quadrilateral ABCD on your notebook, and write E, F, G, H to the midpoints of side AB, BC, CD, DA. What type of quadrilateral is EFGH? // […]. "A quadrilateral connected the midpoints of the each side is a parallelogram." We can prove it as follows. (A proof of the theorem)	By classifying squares, rectangles, and rhombi as parallelograms, it is sufficient to prove the quadrilateral is a parallelogram, not necessary to prove it is a square, a rectangle, and a rhombus.
(iii) A useful conceptual schema during problem solving	[Eighth grade (Problem)] Is the following statement true? If it isn't true, show a counter-example to explain it. // […]. // A quadrilateral with perpendicular diagonals is a rhombus.	By using the property that rhombi are the intersection of kites and parallelograms, we can present a kite as a counter-example.
(iv) Alternative definitions and new propositions	[Not identified] Alternative definitions of quadrilaterals are not explicitly presented in the textbooks, because we cannot give contradictory two definitions to the same concept in a textbook.	
(v) A useful global perspective	[Eight grade (Problem)] In order to a parallelogram becomes a rectangle, a rhombus, or a square, what conditions should we add to parallelogram ABCD? Choose the condition from the following. // $\angle A = 90°$, AB=BC, AC=BD, $AC \perp BD$ (The figure of a process from parallelogram ABCD to rectangle ABCD or rhombus ABCD, and to square ABCD)	We can acquire a useful global perspective about sides, angles, or diagonals. For example, if we add to a condition of equal sides, equal angles, or both to a parallelogram, it becomes a rhombus, a rectangle, or a square.

Table 1. Functions of a hierarchical classification in textbooks of *Tokyo Shoseki* (// means a paragraph brank, […]. means an omission of a sentence)

CONCLUSION

We conclude that not all the functions of a hierarchical classification of quadrilaterals are presented in Japanese textbooks and that there is a limitation that the *only non-contradict definitions* to the same concept must be presented in a textbook. In other words, textbooks can present only a part of defining activities.

This conclusion is based on the result of analysis of only two Japanese textbooks. We need to analyse the presentation of a hierarchical classification of quadrilaterals in other countries' textbooks, and to clarify whether the limitation can be applied not only to Japanese textbooks but also to other countries' textbooks.

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References

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

De Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, *14(*1), 11–18.

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twenty Second International Conference for the Psychology of Mathematics Education* (*Vol. 2*, pp. 248–255). South Africa: PME22.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3–4), 413–435.

Fujii, T. et al. (2016). *Shinpen Atarashi Sugaku 1–3.* JP: Tokyo Shoseki.

Fujita, T., & Jones, K. (2014). Reasoning-and-proving in geometry in school mathematics textbook in Japan. *International Journal of Educational Research, 64*, 81–91.

Jiji Press. (2015). 2016 nendo Chugakko Kyokasho Saitaku Jokyo Monkasho Matome. *Naigaikyouiku, December 8*, 5–7.

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education*, *46*(4), 423–454.

Miyakawa, T. (2017). Comparative analysis on the nature of proof to be taught in geometry: the cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics*, *94*(1), 37–54.

Okamoto, K. (2016). *Mirai he Hirogaru Sugaku 1–3.* JP: Keirinkan.

Usiskin, Z., & Griffin, J. (2008). *The classification of quadrilaterals: A study of definition.* Charlotte, NC: Information Age Publishing.

Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*, 57–75.

THE GENDER TROUBLE IN MATHEMATICS TEXTBOOKS: A COMPARATIVE ANALYSIS BETWEEN BRAZIL AND THE USA

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Textbooks, which still are one of the most used tools in the mathematics classroom worldwide have a substantial impact on the production of students' subjectivities. In this sense, despite research showing that there is a need in eliminating gender bias in mathematics textbooks, they still reproduce the portrayal of what it means to be a girl/woman or a boy/man in our current society. Therefore, this study provides a comparison between one of the most used 6th-grade mathematics textbooks in Brazil and in the USA in order to unveil the ways gender subjects were presented through images and gender citations. This analysis shows that textbooks are still major tools to reproduce stereotypical gender positions for girls and boys, which might influence the creation of students' subjectivities in both countries.

THE GENDER TROUBLE

In 2015, a report by the United Nations Educational, Scientific, and Cultural Organization (UNESCO) advertised the urge of "eliminating gender bias in textbooks." This report analyzed textbooks from different countries around the world and concluded that girls and boys are represented in opposite ways in most of the analyzed materials: girls were shown as passive, kind, partaking in housework activities, and other stereotyped positions for women while "boys and men undertook exciting and worthwhile endeavours and occupations" (Blumberg & Kenan, 2015, p. 01).

Currently, mathematics textbooks also have a significant role in maintaining these gender positions as shown in Neto (2018) and as identified by another UNESCO report:

> Even in some mathematics textbooks, such as those in Turkey, traditional roles among the females in the house (mother, daughter) are portrayed in the context of cooperation. Accompanied by an image of a mother and a daughter cooking together, the following statement illustrates this pattern, "There were four eggs on the table. Ayse brought her mother two more eggs. It summed up to six eggs" (UNESCO, 2016, p. 13).

A set of studies have pointed to textbooks as contributors to the constitution of the subject (see Peñaloza & Valero, 2016), which in turn also applies to mathematics textbooks (see Silva, Valero, Manoel, & Berto, 2018).

In this paper, we conducted a comparative analysis of two of the most used mathematics textbooks for the 6th grade in Brazil and the United States. We investigated how these two empirical materials stylized gender practices in order to go beyond the teaching and learning of mathematics. Simultaneously, we normalized the conduct of

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 281–286). Paderborn: Universitätsbibliothek Paderborn.

the students by means of the desired discourses for gender performances, consequently producing their subjectivity.

Biological information of the sexes (Laqueur, 1990) and the ideas about femininities and masculinities (gender) are not treated as the same in this study. The latter seeing as discursively constructed signs (Richard, 1996), supported by supposed truths that manage and address social bodies practices. In this paper, we perceive the female and male bodies as a result of a set of regulatory social practices (Butler, 2010). Therefore, this study is mobilized in order to describe and analyze the gendered practices that surround both textbooks. For us, mathematics textbooks are important tools that influence the creation of students' subjectivities. We assumed that school mathematics practices, through its contents help, reinforce, justify, and validate values and moralities, which are the contingent notions of nowadays gender understandings. In sum, our point is that mathematics textbooks are a way to teach students about their suitable positions to perform suitable gender stylized practices.

METHODS AND DATA SOURCES

During our comparative analysis, we highlighted the similarities and differences in gender inscriptions in mathematics textbooks from Brazil and the USA.

From Brazil, we examined the textbook *Praticando Matemática* (Andrini & Vasconcelos, 2015) because it is the most used one in public schools. The US textbook chosen was *Glencoe MATH* (Carter et al., 2015) because of its popularity among the most used textbooks throughout the US.

Our methodological approach consisted of the reading of each textbook and selecting reference of genders (professions, occupations, and performed activities), this exercise was done selecting each excerpt about gender and placing into the analysis process. Thus, mathematics tasks, exercises/activities as well as images and drawings were the content of analysis. In our interpretation, the systems of images that surrounded both the feminine and masculine representations weaved a discursive logic that exposed the desirable gender practices of girls/women and boys/men.

FINDINGS

Brazil

Through discrete counting, 155 images and mentions to boys/men compared to 123 references to girls/women were found. Despite the numerical difference (it is important to mention that the boys/men's representation was higher than girls/women's), our gaze is about the gender representation attributes.

For example, regarding professional activities, we identified 24 types of careers for boys/men. Some of them are farmers, showmen, astronomers, singers, managers, drivers, salesmen, fishermen, doctors, teachers, mathematicians, and other activities strongly linked with paid work and sometimes exciting/challenging jobs. Such as, in Figure 1 below, where a group of businessmen are deciding how to share a profit. In this particular case, they needed enough understanding of percentage to make an efficient decision. To know how much money each man received can be determined through some mathematical operations.

6. O gerente de uma empresa recebeu a incumbência de distribuir um prêmio de R$ 12.000,00 entre três funcionários, de acordo com a eficiência de cada um. Se um deles recebeu 20% desse valor e um outro recebeu 55%, quantos reais recebeu o terceiro? R$ 3.000,00

Translation:

6. The manager of a company was given the incentive to share a prize of R$12,000 among three employees, according to the efficiency of each one. If one of them received 20% of that value and another received 55%, how many reals (Brazilian currency) did the third receive?

Figure 1. Andrini and Vasconcelos (2015, p. 237)

In the other hand, girls/women were often represented as seamstresses, school principals, newswomen, doctors, teachers, saleswomen, and secretaries. They were also responsible for their home management, including grocery shopping. In addition, all home cooks were represented by girls/women. Such as observed in Figure 2, where Julia needs to know about fraction knowledge to calculate the necessary butter for her recipe.

Dona Júlia vai fazer um bolo. A receita indica a utilização de um terço de tablete de margarina para a massa e meio tablete de margarina para a cobertura.

- Qual é a quantidade total de margarina necessária?

$$\frac{1}{3} + \frac{1}{2} = ?$$

As frações que devem ser somadas têm denominadores diferentes, portanto representam pedaços de tamanhos diferentes, o que dificulta identificar a fração total resultante. Mas podemos encontrar frações equivalentes a cada uma delas que tenham denominadores iguais. Todos os pedaços ficarão do mesmo tamanho e poderemos contar quantos são.

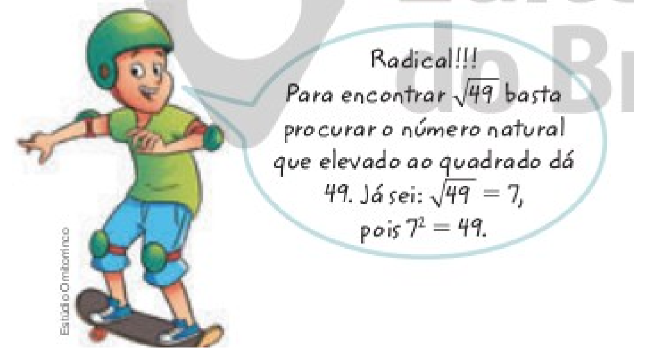
Translation:

Would a stick of butter be enough for this recipe and still have a little left to grease the pan?

Figure 2. Andrini and Vasconcelos (2015, p. 192)

Therefore, we argue that in relation to the social division of labor, the Brazilian textbook still carries strong gender stereotypes which might influence on the way boys and girls see themselves in such texts (UNESCO, 2016). Boys/men were represented occupying many types of professions related to leadership, science, and entrepreneurship, in general socially valued occupations. Most of the professions were strongly connected to business practices by means of neoliberal rationality. While girls/women are shown as kindly, careful, and passive regarding their professions (and sometimes in domestic labor, which often isn't considered a job).

Another important result to highlight is about sports practices. Boys/men appeared practicing different types of sports throughout this textbook, sometimes it happened to contextualize some mathematical content as can be observed in Figure 3:



Translation:

Amazing!!! To find $\sqrt{49}$ just look for the natural number that when multiplied by itself gives you 49. I know it: $\sqrt{49} = 7$, because $7^2 = 49$.

Figure 3. Andrini and Vasconcelos (2015, p. 84)

Meanwhile, girls/women were only represented by practicing hiking. In contrast, boys/men appeared practicing sports in the same proportion as girls/women appeared in shopping activities, such as buying shoes, clothes, etc.

USA

In the US textbook, the binary of the genders was clearly marked by subtle stereotypes in the lines of the textbook. From a binary standpoint, gender was represented relatively equal. There were 230 mentions of girls/women compared to 210 of boys/men. In these citations, names, pronouns, or gender-identified nouns, such as a father or mother, were the unit of analysis. Some of these citations were accompanied by pictures, which in this paper were counted as separate units. A total of 74 pictures portrayed boys/men figures in contrast to 57 portraying girls/women. Therefore, as one browses through the textbook, the male representation is more frequently noticed.

In relation to the professions cited throughout the textbook, there was a noticeable difference. At the end of each chapter, a 21st Century Career was presented to students, together with a picture that represented a specific gender. The careers provided were Cosmetic Chemist (woman), Special Effects Animator (man), Sports Equipment Designer (man), Pastry Chef (man), and Scientific Illustrator in Natural History Art (woman), thus we argue that the diversity of jobs men can do was more than women. Throughout the textbook, women were mostly addressed as an artistic being who do things that “girls do” such as making bracelets, make scrapbooks, or teach. Men were represented as gardeners, athletes, chefs, teachers, musicians, businessmen, construction builders, etc.

Ultimately, one of the aspects noticed in this textbook is the way the conceptualization of boys/men and girls/women were dispositioned throughout the textbook in relation to the “doing of mathematics”. For example, the H.O.T (Higher Order Thinking) problems asserted this way of thinking.

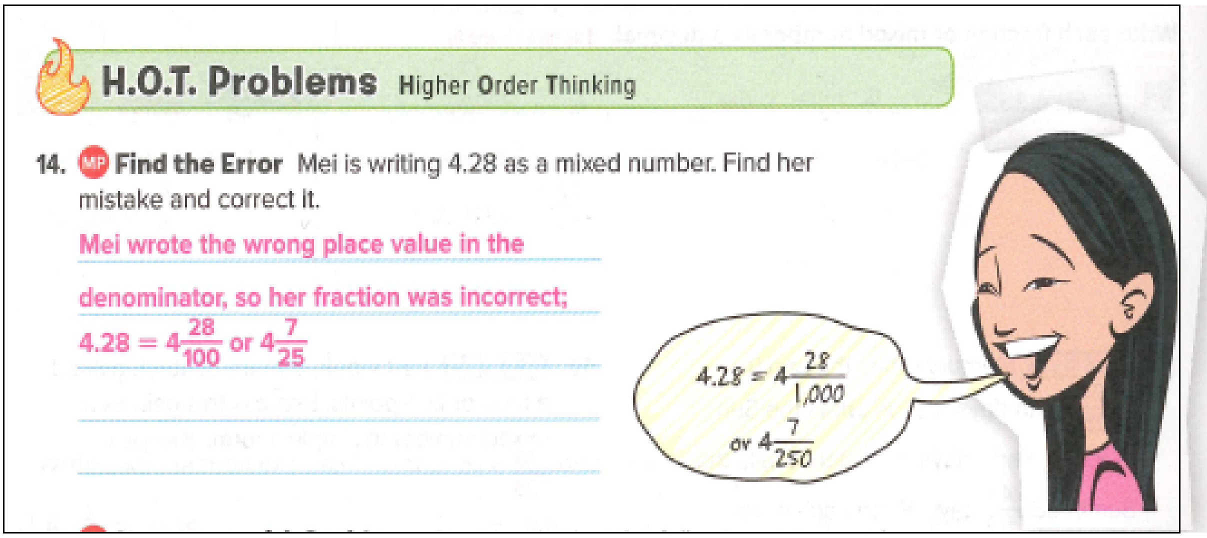
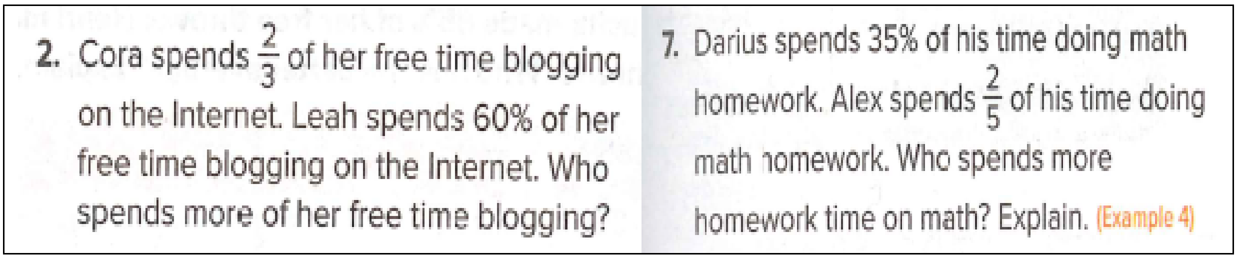


Figure 4. Carter et al. (2015, p. 94)

Most of these H.O.T problems portrayed a student trying to solve a mathematical problem, without success. One of the sub-questions in a H.O.T problem was to find the student's mistake and correct it. Out of the ten times, a H.O.T problem referred to find the mistake of a student, seven times were girls/women' mistakes, which influences in the way we think about girls/women making more mistakes in math when compared to boys/men. Throughout the textbook, an attempt to balance the gender representation is clearly observed, but the textbook fails to equitably show that both genders make mistakes in math. This lack of equitable mistake representation leads us to a biased creation of the gendered subjects (Blumberg & Kenan, 2015).


2. Cora spends $\frac{2}{3}$ of her free time blogging on the Internet. Leah spends 60% of her free time blogging on the Internet. Who spends more of her free time blogging?

7. Darius spends 35% of his time doing math homework. Alex spends $\frac{2}{5}$ of his time doing math homework. Who spends more homework time on math? Explain. (Example 4)

Figure 5. Carter et al. (2015, pp. 132–133)

In addition, boys/men were always dispositioned as the leader of the conversation, while girls/women were displayed on the background serving boys/men.

LAST REMARKS

This research concluded that in both textbooks, there was a specific societal belief implied to the different genders displayed throughout the text. Boys/men represented in these textbooks still carries the ideas of a patriarchal performative society, in which men are strongly related to the ideas of power which contributes to a stereotypical gender idea of men. Women representation were backgrounded and envisioned as "the support being," reinforcing ideas of stereotypical genders (Neto, 2018). Particularly, in the US textbook, girls/women are usually depicted as subpar when compared to boys/men in "doing mathematics." A similar situation happens in textbooks from

Brazil with the addition of the women are often being represented in labor activities that are less valued socially and less challenging, which in turns strengthen both gender bias and the creation of gender stereotypical subjects (Blumberg & Kenan, 2015).

Despite the emergency to eliminate the gender stereotypes in math textbooks, little research has been done with such emphasis (Fan et al., 2013). In the same line, our analysis shows evidence for the necessity to continue deepening the investigations with this focus in order to combat the gender inequalities in all spheres of society, including mathematics textbooks that current still represent stereotypical gender roles.

References

Andrini, A., & Vasconcelos, M. J. (2015). *Praticando matemática* (4th ed. Vol. 6). São Paulo: Editora do Brasil.

Butler, J. (2010) *Problemas de gênero*: feminismo e subversão da identidade. Rio de Janeiro: Civilização Brasileira.

Carter, J. A., Cuevas, G. J., Day, R., Malloy, C., Kersaint, G., Reynosa, M. E., Zike, D. (2015). *Glencoe math: Built to the common core: Course 1.* Colombus, OH: McGraw-Hill

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education*, *45*(5), 633–646. doi:10.1007/s11858-013-0539-x.

Laqueur, T. W. (1990) *Making sex:* Body and gender from the Greeks to Freud. Cambridge: Harvard University Press.

Peñaloza, G., & Valero, P. (2016). Las ciencias naturales escolares y la fabricación del ciudadano católico en Colombia. *Educação Unisinos*. *20*(1), 3–13.

Richard, N. (1996) Feminismo, experiencia y representacion. *Revista Iberoamericana.* Vol. LXII, Niums, Julio-Diciembre, 176–177.

Silva, M. A., Valero, P., Manoel, C. A. L. C., & Berto, L. F. (2018). Brazilian high school mathematics textbooks and the constitution of the good student citizen. *Acta Scientiae*. Canoas – RS, *20*(6), 1701–1801.

Neto, V. F. (2018). Livros Didáticos de Matemática para o Campo: Aqui também se aprende a ser Mãe. In *VII Seminário Internacional de Pesquisa em Educação Matemática*, 2018, Foz do Iguaçú. Anais VII Sipem.

Blumberg, R. L., & Kenan, W. R. (2015). *Eliminating gender bias in textbooks: Pushing for policy reforms that promote gender equity in education.* Paper commissioned for the EFA Global Monitoring Report 2015, Education for All 2000–2015: achievements and challenges.

UNESCO (2016). *Textbooks pave the way to sustainable development.* Retrieved from https://unesdoc.unesco.org/ark:/48223/pf0000246777.

THE NARRATIVE OF SETS OF LEARNING RESOURCES AS DESCRIBED BY TEACHERS

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Teachers are central actors in the interpretation and implementation processes of curricular resources. This study focuses on means of communication that teachers use to describe the narrative, underlying main characteristics, of a set of tasks. The study explores which characteristics do teachers use when they characterize through tagging a set of tasks of their own choosing, and how does their description of the sets' narrative use different levels of analysis of the tagged characteristics. Participants were 4 individual teachers in a professional development program during a graduate school course. The course focused on analysis of textbooks and curricular materials, and included experimenting in tagging and representing different sets of learning resources, such as tasks, interactive diagrams, lesson plans, etc. The tagging included choosing the set of resources and characteristics to be tagged, and then the use of these characteristics to describe the narrative of the chosen set of learning resources.

BACKGROUND

Teachers interact with learning resources on a daily basis, and in various stages of their work. Whether it is in their professional development, lesson preparation, and also during classroom instruction and student assessment. There are many occasions in which teachers do not interact with a single learning resource, or just a small number of them in a designed setting (lesson, assessment). In some cases, teachers need to gain an understanding of a whole textbook, or in the digital age a repository of learning resources hosted on a website (Olsher & Yerushalmy, 2018), and different teachers hold different perspectives about curricular materials.

Conveying a textbooks voice, or goals of a set of tasks, is not a trivial task. Remillard, Reinke, and Kapoor (2018) demonstrate potential added values in communicating goals of lessons in depth in textbooks as means to assist teachers to steer towards these goals better in their teaching. Olsher and Even (2018) found that teachers expressed a need for organizing tools to increase the accessibility of textbook contents when performing modifications to textbooks they use in the classroom, thus not settling for the standard table of contents and an elaborated teacher's guide.

Recent studies examine features of teacher's resource work, e. g. the Documentational Approach to Didactics (Guedet & Trouche, 2009), and many artefacts are developed in order to assist teachers reflect on the resources they interact with, one example is the "*Reflective Mapping of Resource System (RMRS)*, in which a teacher is asked to draw a map to present her resources in a structured way based on her own reflection" (Wang, 2018). These constructs support the analysis of teacher's interaction with learning

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resources in different stages of their teaching, and shed more light on different perspectives on curricular design and emphasis between mathematics teachers and mathematics education researchers, as these categories are negotiated (Cooper & Olsher, 2018). These differences suggest both challenges and opportunities for teachers' productive use of these tools.

In this preliminary study, I take another step towards creating a common ground between teachers and mathematics education researchers, as well as among teachers, when interacting over curricular resources. Asked to describe a collection of learning resources, namely tasks in this case, teachers defined the relevant characteristics of interest to them, and then analyzed the data in order to tell the story of the collection of learning resources. By doing this, teachers provided an opportunity to study what they view as important to tell about collections of learning resources, and about their ability to supply meaningful information using tagging and visual representation analysis (Olsher & Yerushalmy, 2018).

METHOD

Research aim and questions

The research aim of this study is to examine how teachers describe an underlying narrative of a set or repository of learning resources. In order to achieve this, I address the following research questions: 1. What are the characteristics that teachers choose in order to describe (tag) a set of learning resources? And 2. How does the teacher's description make use of the tagged characteristics information?

Research setting

This study analyses the tagging and descriptions of 4 individual teachers as part of a professional development course during their graduate studies. Two of the teachers teach in secondary school, and two teach in elementary school. The teacher's years of experience ranged between 2 and 6 years of teaching. The course focused on analysis of textbooks and other curricular resources, as well as research about curriculum development and enactment. The tasks along the course included experimenting in tagging and representing different sets of learning resources, which ranged from tasks, supplementary resources (e.g., multimedia files), to full lesson plans. This study focuses on the final assignment of the course as the data source. The assignment was composed of three parts: 1. Choosing 50 different resources (e.g., tasks), 2. Selecting the characteristics to be tagged, and tagging the chosen resources, and 3. Use the chosen characteristics to describe the narrative of the set of learning resources.

The teachers used a pair of tools developed for the studying the balance and inter-relationship between sets of learning resources comprising textbooks or other repositories (Olsher & Yerushalmy, 2018). The tools are a browser extension for assigning different characteristics of learning resources, and a web-based dashboard-configuration tool (https://keshif.me/) to view and analyze relationships between tagged learning resources from a certain textbook or repository. One of the uses that I envision for these tools is that teachers will be able to use these tools in order to analyze a collection of learning resources, providing them insights and assisting them in describing the voice or narrative of the tagged collection.

Data sources and analysis

Data sources include the selected set of characteristics chosen by each one of the teachers. Some of the characteristics were categorical (e.g., role – introduction, homework, etc.) and some were numeric: describing the level of this characteristic (e.g., a scale between 0–3). Another data source were spreadsheets including the tagging of the different teachers for their selected learning resources. The four teachers in this study selected sets of tasks as their resources for description. Each spreadsheet includes a record for each resource, and then the different tagging for the resource according to the characteristics chosen by the teacher. Apart from the spreadsheets each teacher submitted a written analysis of the curricular resources, that was aimed at describing the certain attributes of the selected set according to the tagged characteristics. The teachers referred to this description as "the story of the set". The first level of analysis of the data was informed by categorizing the different characteristics of tagging selected as characteristics of coherence of design or coherence in use (Pepin, Gueudet, Yerushalmy, Trouche, & Chazan, 2015). The second level of analysis coded how the teachers used these different characteristics in their analysis, in order to better understand their use of the tagged data.

RESULTS

Findings show that two of the teachers chose to describe a collection of learning resources they did not use in their classroom (e.g., a section of an e-textbook), while the other two teachers chose to describe their own teaching sequence of a specific topic.

Teacher	Tagged characteristics by type of coherence		Characteristics used in narrative description	
	Of design	In use	Of design	In use
Anna	7	1	7	1
Beth	2	0	2	0
Carly	4	2	4	2
Dennis	4	2	1	1

Table 1. Number of characteristics of coherence of design and coherence in use used by the different teachers

As shown in Table 1, Three out of the four participants used both coherence of design and coherence in use characteristics in their description. Beth (pseudonyms are used for all of the participants), which was one of the teachers who did not describe her own teaching sequence, was the only one to use only coherence of design characteristics in her analysis and description of the collection. In the following sections I describe the different choices of the teachers, and for each teacher I will describe the use of the data in their description.

Anna described a set of learning resources for teaching negative numbers, which she did not previously use in her classroom. She did not specify any specific narrative she was looking to describe in this set. The characteristics chosen were from both the coherence in design and coherence in use. Coherence of design characteristics were: sub-topics, drawing conclusions, technology use, representation, type of resource (video, text, mathematical aid, dynamic diagram), and level of difficulty. The only coherence of use characteristic was student assigning (whole class, groups, pairs, independent work). Anna's description of the set included statistical facts that she studied using the dashboard representations of the data, e.g. "64% of the tasks has symbolic representation, 16% with verbal representation, 22% with numeric representation, and 14% with graphical representation". Anna's description did not articulate further. She did not share any conclusions that could be drawn from the results she discovered in the dashboard.

Beth described a set of learning resources using a single characteristic: cognitive demand. The characteristic that was chosen was a coherence of design characteristic, which was also used in the description. Thus this analysis is de-contextualized, and gives an "objective" perspective on this set of learning resources. Beth's aim was to analyze a chapter in a textbook that teaches linear functions. The author of this textbook claims that the teaching sequence of the different 6 sub-chapters of this topic could be mixed. Beth wanted to check if this was still relevant if one believes that teaching should start with lower cognitive demand tasks for the first topic, and then gradually raise the amount of higher cognitive tasks demand. Her analysis led her to discover that there were different percentages of low cognitive demand tasks in each sub-section. Beth used this information to conclude that if a teacher wishes to teach starting with relatively "easier" tasks in the sub-section, then it is advised that he will progress in a specific trajectory, rather than choose any of them. While this assumption might be questioned - Beth performed a focused analysis that served her in describing the set of learning resources, used the information she gathered in order to draw conclusions, and informed her decision making about enactment of this set of tasks.

Carly's aim was to analyze her own teaching of fractions in the fourth grade, and to understand to which level does she use different types of resources: aids, illustrative measures, digital means. The characteristics chosen were from both the coherence in design and coherence in use, as Carly chose to describe a set of enacted tasks in her classroom, in order to reflect on her teaching. Coherence of design characteristics were: mathematical actions, arithmetic operations [on fractions] (addition and subtraction, and comparison). Representation, Type of resource (realistic story, Illustrative measures, Technological tools). Coherence of use characteristics were: Role (exercise, opening tasks, assessment, homework, enrichment), Student assigning (whole class, groups, pairs, independent work). Carly's analysis led her to choose the

Role of the task as a leading characteristic for her narrative, describing the different sub-sets of tasks divided into their roles. Her description of the coherence in design characteristics remained brief and informative, yet justifications were made to the different types of resources used for different roles:

> Assessment tasks are aimed at checking if the students understood the topic. Thus, in these tasks there is no use of mathematical aids, technological tools, or daily life stories. Because this is a more advanced stage in which the expectation is that students are supposed to solve exercises with fractions without using tools (Carly, final assignment).

The summary of the narrative concluded that:

> the reason for the appearance of different types of resources this way is that the need for these resources depends on the goal of each of these different roles: opening, homework, enrichment, assessment, practice, and the types of tasks" (Carly, final assignment).

Like Beth, Carly performed a focused analysis that served her in describing the set of learning resources, used the information she gathered in order to draw conclusions or support her decision making when choosing the different resources.

Dennis' aim was to analyze her own teaching of fraction comparison in the fifth grade of low-achieving students. The characteristics chosen were from both the coherence in design and coherence in use, as this teacher chose, like Carly, to describe a set of enacted tasks in her classroom in order to reflect on her teaching. Coherence of design characteristics were: mathematical actions, arithmetic operations [on fractions] (comparison, verbal description, problem solving), Representation, Type of resource (mathematical aid, technological tools, stories, notebook tasks, worksheet, blackboard tasks). Coherence of use characteristics were: Role (explanation, practice, homework), and student assigning (whole class, pairs, independent work).

Dennis' analysis was divided into the use of technology in the different tasks' roles. Although there is a significant difference in the percentage: technology is used in 100% of the explanation tasks, but in less than 50% of the practice tasks, even though the teacher does not indicate any reason for this difference and simply states: "using technological tools in each of the stages raises the motivation of the lower achieving students for learning". Dennis describes the different results she discovered from the dashboard, e. g. "there were 7 explanation tasks (14%), 27 exercise tasks (54%), and 13 homework tasks (26%), without going into further description or explanation. Like Anna, Dennis' description did not articulate further. She did not share any conclusions that could be drawn from the results she discovered in the dashboard.

DISCUSSION

The teachers chose different types of characteristics when tagging the set of learning resources. These findings are consistent with the fact that different teachers have different perspectives when it comes to curricular materials.

Two of the teachers used the results extracted from dashboard in order to draw conclusions and the respective collections. The other two teachers chose a descriptive approach and settled for just presenting the different numeric relations between characteristics from the dashboard, without further analysis. One possible explanation for the teacher's descriptive approach is the fact that they were not able understand or to explain the meaning of the results. Future research should take into consideration that teachers might lack the relevant knowledge required to interpret the relational information displayed by the Keshif browser, and remain on the level of raw results.

While three of the four teachers included at least one coherence in use characteristic in their analysis, the remaining teacher did not address any coherence in use characteristics, and attempted to provide an "objective" analysis of a textbook – regardless of the context of its enactment. This objective perspective presented by a teacher is not aligned with previous studies (Cooper & Olsher, 2018) that suggest that teachers tend to emphasize contextual characteristics.

Acknowledgements

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References

Cooper, J., & Olsher, S. (2018). Boundary crossing in design based research—Lessons learned from tagging didactic metadata. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 299–306). Umeå, Sweden: PME.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199–218.

Olsher, S., & Even, R. (2018). Organizing tools suggested by teachers in the mathematics textbook they use in class. *Journal of Science and Mathematics Education*. Advance online publication. https://doi.org/10.1007/s10763-018-9902-2

Olsher, S., & Yerushalmy, M. (2018). Making informed decisions: Teachers' interacting with curriculum resources. Presented at ICMI study 24 (Japan, November, 2018).

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. I. (2016). E-Textbooks in/for teaching and learning mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education: Third Edition* (pp. 636–661). New York: Routledge.

Remillard, J., Reinke, L. T., & Kapoor, R. (2018). What is the point? Examining how curriculum materials articulate mathematical goals and how teachers steer instruction. *International Journal of Educational Research*. Advance online publication. https://doi.org/10.1016/j.ijer.2018.09.010

Wang, C. (2018). Mathematics teachers' expertise in resources work and its development in collectives. A French and a Chinese Cases. In L. Fan, L., Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on Mathematics textbooks and teachers' resources: Advances and issues* (pp. 193–213). Cham: Springer.

COMPARING MATHEMATICS TEXTBOOKS — AN INSTRUMENT FOR QUANTITATIVE ANALYSIS

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New national curricula and another perspective on mathematics education like for example in the New Math in the 1960th make textbooks change. Thereby, not only the mathematical content differs, even more the presentation and handling of mathematics show great varieties. To identify and explore such differences with statistical methods, a category system for analysing textbooks was developed. By now 30 000 data sets of 14 textbooks are collected. Thereby, first assumptions about differences and changes were made.

INTRODUCTION

Mathematics textbooks vary in many ways. For example, they use different kinds of tasks, reaching from easy technical exercises to such requiring mathematical argumentation or proof. Some textbooks are overloaded with pictures not showing any use for learning mathematics, while other books use images to explain and illustrate mathematics. Great differences can also be seen in the way of presenting mathematics. While some textbooks present a minimal amount of information or contain only a few worked examples, others have long texts, explaining mathematics, or they have special tasks, advising students how to handle a single mathematical problem. To outline such differences, being found in various decades or textbooks of different countries and publishers, an instrument was developed. First of all, this article provides an overview of the research questions and the category system. Afterwards, first hypotheses from an exemplary study are described.

AIMS

Often, the research on mathematics textbooks focuses on very special aspects like the use of materials to show mathematical structure (cf. Dooley, 1961, p. 317) or on aspects "easy to quantify" (translated from Reichmann, 2008, p. 331, „leicht zu quantifizierende Daten“). Thereby, isolated aspects of mathematics textbooks, like the number of images (e.g., Reichmann, 2008; Dooley, 1961) or the use of exercises instead of tasks (Valverde et al., 2004) are regarded. On the other hand, historical studies often use qualitative methods that show very detailed information. But they don't give a hint on the relevance of a change because differences are only indicated, while the frequency of an aspect is unconsidered.

In consequence of the difficulties shown above, three aims were settled for the following study:

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1. To avoid considering only a few, maybe special, aspects of textbooks some basic categories for analysing textbooks had to be found. Therefore, results from textbook research were used as well as findings about the three main textbook elements (images, explanations and tasks). Additionally, a qualitative study on the base of 30 German textbooks from different decades and school types was done to complete the category system.

2. To decide if differences between books of distinct countries, decades, publishers or school types are significant and to examine what is typical for a decade, the instrument should produce quantitative data. As a result, statistical methods can be used.

3. Finally, not only isolated data should be collected. Instead, coherencies among categories should be considered. So, for example, the correlation between the type of explanations and the mathematical content can be shown.

CATEGORY SYSTEM

Considering these aims, in this study an instrument was developed to measure differences between textbooks by statistical methods. The category system focuses on three main textbook elements: images, explanations and tasks. These can be seen as a conclusion of the more detailed "blocks" by Valverde et al. (2002). To get detailed and adequate statistical characterization of these three elements, from seven up to 17 categories were defined for each element. Every category is specified by properties. Overall, there are around 140 properties that can be analysed individually or related to other properties. To get quantitative data, every textbook element in one textbook is analysed by the categories of the category system. Counting how many times a special property occurs leads to a quantitative set of data that can be handled with statistical methods. Having mainly nominal scaled data chi-square methods are used.

APPLICATIONS

14 textbooks were analysed to show in an exemplary way how the instrument can be used and to get first impressions about changes in German textbooks. These books are published in the years 1935 to 2011. To make changes more clearly, these books can be summarized in five exemplary eras, beginning with the time of the National Socialism in the late 1930s and early 1940s. Because there are no changes in national curricula, the textbooks of the 50s and 60s are considered as one era. They are followed by a special period, the New Math in the late 60s and 70s. Afterwards the 80s and 90s are treated as one era, belonging to the same curriculum. The last era, based on the curriculum of 2004, is resulting from books published between 2004 and 2015.

Data base

To get information, mainly about historical variances, as many as possible variables of the chosen textbooks were kept constant. So, every textbook is published in Bavaria (a traditional federal state in Germany). All books were produced for the 7th grade of the Hauptschule (the lowest school type in the German tripartite school system). In every era at least two textbooks from established textbook publishers were selected (Buchner/Klett, Oldenburg, Ehrenwirth). Wherever possible, the same textbook series (e.g. "Wir rechnen", "Die Welt der Zahl", "Formel") were analysed in different eras.

The textbooks were surveyed completely. That means, not only single chapters of the books were analysed, but every textbook element (image, explanation, task). That leads to a data set of 30 000 textbook elements. Having only a small range of textbooks analysed, no absolute results can be given despite that great number of records. Nevertheless, hypothesis can be shown by the present set of data.

Generic issues

First of all, the collected data can be used for frequency analysis. As a result, it is possible to get information about **number or amount of a special property**. For example, 95 tasks in the three analysed textbooks of the 2000s require mathematical argumentation, which is 3 % of all tasks of this era. 1257 tasks have an extra-mathematical content (in average 419 per textbook or 41 % of all tasks in this era). Looking at the category "type of explanations", three properties are regarded. First, mathematical operations can be explained by *worked examples*. Another way to give information to students is in an explicit, narrative way, here called *narrative explanations*. Finally, information can be reduced on short conclusions of important formulas. This is called *short explanation*. In the textbook "Formel 7" (produced in the year 2004) 61 % of the explanations contain *worked examples*, 33 % *short explanations* and 6 % *narrative explanations*.

To get more detailed insights, a **comparison** of different textbooks is helpful. Comparing the distribution shown above with other publishers' textbooks or another edition of "Formel 7", you can figure out whether this distribution is typical for textbooks of this era. Looking exemplarily at "Mathe aktiv 7" (2005) no significant difference can be established, concerning the "type of explanation". What's more, comparing "Formel 7" (2004) with the following edition, published in 2011, no significant difference can be seen. That's not surprising, knowing that 65 % of the explanations in the newer book are printed in the old one in exactly the same way. This finding should be affirmed by analysing more textbooks from other publishers and other grades. While this category is typical for the publishing period, other aspects of textbooks depend on the respective author or publisher. So it seems, that the number of images used to help the students to orientate in the textbook, depends mainly on the publisher. While in the textbook "Formel 7" (2004) 16 % have this function, in the textbook "Mathe aktiv 7" (2005), there is no such image.

A particular type of comparison is the presentation of distinctions of textbooks published in different eras. Doing that, varieties in textbooks depending on the publishing period can be detected. Looking at the "type of explanation" in different eras, a significant change can be seen:

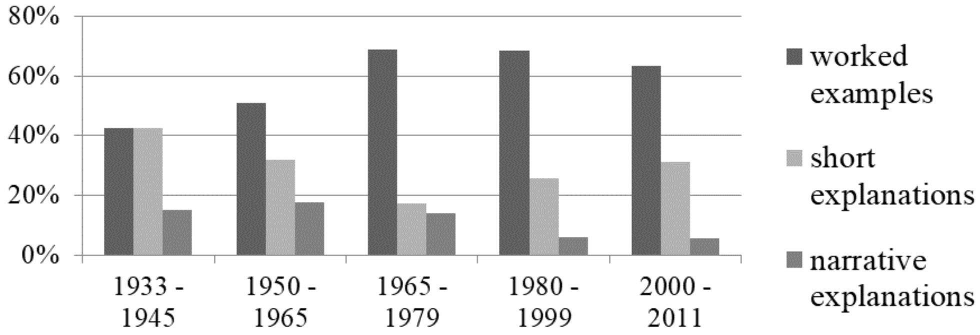

Figure 1. Distribution of the category “type of explanation” in different eras

Another finding, seen from the comparison of different eras’ textbooks, is the image type *illustrated diagram*. Occurring mainly in textbooks published in the 50s and 60s, this property can be regarded as typical for this period. Detecting those generic aspects of special eras is another application of the instrument.

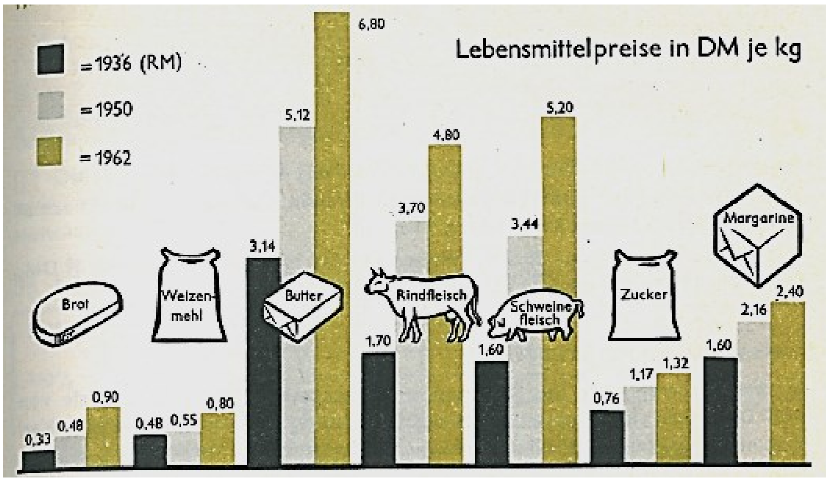

Figure 2. Image with the property *illustrated diagram (*Schlagbauer, 1964, p. 41)

One of the instruments’ aims was to enable **measuring coherencies**. That is shown on the example of textbook images. Maybe someone is not only interested in the “types of images” used in a special textbook, but rather in differences in the use of these image types. In general, images are used in different ways. The most important function of images is to *present mathematics* or to *make a text easier to understand*. Other images don’t bring any information, but are used to *illustrate* or to *help the students to orientate* in the textbook. Furthermore, images *present information* needed to solve tasks. In the present data a middle coherency between type and function of images can be proved (Cramer’s V .39). In particular, images used to *help the students to orientate* in the textbook can be presumed, being of one single type. Therefore, only *drawings* of fantastic or real things are used. Further on, there is a middle coherency between these *drawings* and the use as *illustration* (Cramer’s V .44). *Tables* are often used to *present information* needed for tasks (Cramer’s V .38), but not as *illustration* (Cramer’s V .19). In the table below the number of the main image types and their function can be seen.

	diagrams	geometric drawings	drawings	tables	others
presenting mathematic	0%	4%	2%	1%	5%
make texts easier to understand	1%	7% +	3%	1%	2%
present information for tasks	3%	14%	8% -	20% +	10%
illustration	0%	0% -	10% ++	0% -	2%
others	0%	0%	6%	0%	1%

Table 1. Coherencies between the categories "type of image" and "function of image" (++/-- means there is a middle coherency (Cramer's V > .40) appearing more/less than expected, +/ - means a weak coherency (Cramer's V > .20))

Many other coherencies can be analysed by the instrument. Resulting on the present data, a weak coherency can be stated exemplarily between the "type of explanation" and the "mathematics content" (Cramer's V .24). While *geometric* explanations use *short explanations* mostly (55 % of the geometric explanations), explanations about *numbers and calculation* prefer *worked examples* (74 % of the explanations of this topic).

In the previous applications, the structure of the category system was used to come to new questions. In addition, **further questions** can be surveyed by the help of the instrument. For example, the effect of pedagogical, didactical or mathematical influences on textbooks, like the implementation of national standards or the results of PISA or TIMMS, can be analysed. Therefore, the issues have to be translated into categories or properties analysed in this instrument. This is exemplified for the New Math in the late 1960s. First, basic demands on textbooks had to be found. According to Hayen (1987) one of the demands for modern textbooks in the New Math period was another way of giving information to students, meaning "many examples" (translated from Hayen, 1987, p. 113) and no "reading book for pupils" (Hayen, 1987, p. 115). Translated in the categories of the instrument, the "type of explanation" and the property of tasks *advising students how to handle a single mathematical problem* can be used to find an answer. Solely looking on textbooks of this era will help less. More important is the comparison of New Math books with further editions to see varieties. At first sight, Hayen's demands seem true for the present data. As shown in figure 1, number and amount of *worked examples* increase in the New Math era. Also, the amount of *narrative examples* decreases. Nevertheless, the actual number of this property rises. To get more differential findings, the two analysed textbooks can be regarded separately. Comparing the distribution of the "types of explanation", no significant difference between the two books can be proved. A different point of view arises from the comparison of "Gamma 7" (1978) and its previous textbook "Wir rechnen 7" (1964). Contrary to the demands, the number of *narrative explanations* increases. Reasons may be different, but they are not part of the current study.

Special tasks *giving advice how to handle a single mathematical problem* are used in the textbook “Gamma 7”. They present mathematics by concrete activities, guiding students to solve a mathematical problem. Having an amount of 12 % (of all tasks), this finding could support Hayens’ demand for no “reading book”, especially as this property increases in the New Math.

In the same way other characteristics of the New Math can be surveyed. Exemplarily, there is a significant increase of images, using colours to show the mathematical structure and not just make the book more colourful. Rising from 7 % up to 33 % in the New Math and decreasing in the following era, the useful assignment of colours in images can be seen as a property typical for the New Math era.

CONCLUSION

As shown above the instrument is able to show changes in textbooks in different eras. By now, there are only ideas about these changes. To assure those first results, analysing an adequate sample of further textbooks is necessary. Though the instrument produces quantitative data, there are very detailed findings about the way mathematics textbook are structured. This depends on the developed category system, considering a great number of aspects. Some of these categories produce information on their own, or they can be combined to get more detailed information about presentation and handling of images, tasks and explanations. As shown above there are very different ways of using this instrument and many questions about textbooks can be answered.

References

Bauhoff, E., & Wynands, A. (Eds.). (2005). *Mathe aktiv 7. Mathematisches Unterrichtswerk für Hauptschulen.* Braunschweig: Schroedel.

Dooley, C. (1961). The Relation between arithmetic research and the content of arithmetic textbooks, 1900–1957. *The Journal of Experimental Education, 29*(3), 315–318.

Hayen, J., Vollrath, H., & Weidig, I. (Eds.) (1978). *Gamma 7. Mathematik für Hauptschulen in Bayern.* Stuttgart: Klett.

Hayen, J. (1987). *Planung und Realisierung eines mathematischen Unterrichtswerkes als Entwicklung eines komplexen Systems*. Oldenburg: Klett.

Schlagbauer, A., & Rabenstein, R. (1964). *Wir rechnen 7*. Bamberg: C. C. Buchner.

Reichmann, K. (2008). Das Schulbuch im Mathematikunterricht – Entwicklungstendenzen zwischen 1870 und 2000. *Der mathematische und naturwissenschaftliche Unterricht*, *61*(6), 326–332.

Sailer, W., & Vollath, E. (Eds.) (2004 & 2011). *Formel 7. Mathematik.* Bamberg: C. C. Buchner.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book—using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer.

HISTOS — TEACHERS' CONNOTATIONS ABOUT HISTORICAL SNIPPETS IN TEXTBOOKS

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Interactions between teachers and textbooks are an active process and have an impact on mathematical lessons. International studies show that historical snippets are one way to include the history of mathematics in textbooks. Teachers must handle these tasks, but in which way do they use these historical snippets? The aim of this explorative survey is to identify teachers' language about tasks concerning the history of mathematics. Which connotations do teachers have about these textbook tasks? Do teachers identify the same benefits of the history of mathematics in education as researchers do? This paper proposal will give an insightful view of a pilot study on prospective teachers' language about historical snippets.

INTRODUCTION

Besides a various field of teaching resources, textbooks play a crucial role in most mathematical lessons (Howson, 1995; Hiebert et al., 2003). Following, "textbooks" will be understood as developed for students' consumption and include problems, texts, exercises and more (Fan, Zhu, & Miao, 2013). In addition to the interaction of textbooks and students, there is the teacher and the mathematical content. Between the four elements—textbook, student, teacher, and mathematical content—there is an interdependent linkage. The following survey concerns the interaction between textbook tasks and teachers within a mathematical historical content.

According to Gueudet and Touche (2009), teachers use resources, in particular textbooks, to design their lessons and transform resources depending upon their needs. Remillard (2018) argues that the interaction between teachers and resources depends on affordances of resources and interpretation of teachers. The concept of pedagogical design capacity (PDC) helps to understand this interaction between teachers and resources. PDC is defined by Brown (2009, p. 29) as a "teacher's capacity to perceive and mobilize existing resources in order to craft instructional episodes". Therefore, interactions between teachers and textbooks are an active process of re-sourcing the resource (Adler, 2000, p. 207) and have an impact on mathematical lessons. According to Hersh (1979, p. 33): "One's conceptions of what mathematics is affects one's conceptions of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it." If we consider that tasks concerning the history of mathematics are included in textbooks and that teachers use their PDC to interact with these tasks, which connotations occur in teachers' interpretations of textbook tasks about the history of mathematics?

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TASKS ABOUT THE HISTORY OF MATHEMATICS IN TEXTBOOKS

The history of mathematics in textbooks pursues different goals. There are various arguments for the benefits of the history of mathematics in education. Bütüner (2018) describes a four-item list of benefits in a synopsis of his research (cf. Fried, 2001; Liu, 2003; Tzanakis & Arcavi, 2002):

> (ɪ) It may help students understand that mathematics is a human activity and product […] (ɪɪ) It may help raise students' motivation and develop a positive attitude towards learning […] (ɪɪɪ) It may develop student perspectives on the nature of mathematics and mathematical activities as well as teachers' instructional repertoire […] (ɪv) It may help the understanding of mathematical concepts, problems and solutions […]. (Bütüner, 2018, p. 9)

The international overview shows only small variations on types of tasks concerning history of mathematics. For example, Xenofontos and Papadopoulos (2015) identifies two types of tasks in Greek and Cyprus textbooks: those which inform students about the history of mathematics and those which lead to students' mathematical activity. In German textbooks, Schorcht (2018a, p. 155) identifies five types of tasks: "1. informative present, 2. acting present, 3. informative past, 4. acting past, 5. personalization type." Aside from the personalization type, these types can be summarized as the informative and the acting types of tasks. In comparison with the results of these studies, I determine three types of tasks for the survey: informative type, acting type and personalization type.

For the survey, I use tasks developed as "historical snippets" (Tzanakis & Arcavi, 2002, p. 214) to stimulate prospective teachers' connotations, due to the fact that these can most often be found tasks in textbooks (Schorcht, 2018b). Tzanakis and Arcavi (2002) observe historical snippets in two aspects: format and content. A historical snippet has a position before, after or alongside mathematical content. It merely informs students or stimulates students' activity. The detailedness ranges from simple dates into specifics about the history of mathematics. Style and design of historical snippets are different, according to Tzanakis and Arcavi (2002). The possible content of a historical snippet could be, for example, biographical dates, photographs, introduction into topics, anecdotes, etc. The authors describe the conceptual issue as follows:

> the narrative may touch upon motivation, origins and evolution of an idea, ways of noting and representing ideas as opposed to modem ones, arguments […], problems of historical origin, ancient methods of calculation, etc. (Tzanakis & Arcavi, 2002, p. 214)

REPERTORY GRID OF TEACHERS CONNOTATIONS

For the presented survey, the repertory grid methodology seems to be a valuable tool to record data about teachers' connotations on historical tasks in mathematics textbooks. Repertory grid methodology was introduced by Kelly (1955) as a psychological assessment tool. Interviewed persons should generate a list of possible items of interest. After that, the interviewer shows the test persons three items from their list.

They should name the similarity of two items and describe the third one in a different way. These pairs of attributes contain a characteristic and its opposite, and this gives an insightful view on the beliefs in relation to the context. Bruder, Lengnink, and Prediger (2003) used mathematical tasks to explore prospective teachers' language. They execute a pre- and post-interview in their course to elicit a repertory grid. They specify the items for their test to receive more or less comparable results and compared the diversity of attributes in the pre-interview situation to the diversity in the post-interview situation. With this approach, they identify a brief extract of prospective teachers' beliefs for the purposes of evaluating their courses.

THE PRESENTED TASKS

The survey includes an online interview on the basis of repertory grid methodology. For example, one task presents Sofia Kowalewskaja and her biography. The task says:

> Sofia Kowalewskaja was born in Moscow in 1850. Her parents and teacher recognized Sofia's mathematical talents at an early age. Since girls could not attend the university in Russia, she went to Germany when she was 20 years old. However, there were also many obstacles for women in Germany. Sofia did not allow herself to be discouraged and continued her struggle for equal rights. In 1884, she became the first female professor of mathematics in Sweden. (Wittmann & Müller, 2011, p. 115; author's translation)

There are three questions following to prompt the student into mathematical action: "(a) How old was Sofia Kowalewskaja?", "(b) At what age did she become a professor?", "(c) In which year did she move from Russia to Germany?" (Wittmann & Müller, 2011, p. 115; author's translation).

A second task explains a short episode of the history of measurement in Europe:

> Throughout history there have been different units of measure for weights, e.g. Carats, ounces, quints, pounds and hundredweight. The units were different in size depending on time and region. There were, among others, the Roman pound (327 g), the pound of Charlemagne (about 409 g), the Paris pound (789 g) and the Prussian pound (468 g). Only in 1875 was uniformly determined: 1 pound = 500 g, 100 pounds = 1 hundredweight. A hundredweight equivalent to 50 kg. (Becherer, Gmeiner, Hübscher, Huschens, & Schulz, 2010, p. 71; author's translation)

Two questions follow: "Which problems resulted from the old weight units?" and "Can you translate in grams and kilograms?" (Becherer et al., 2010, p. 71; author's translation) The first question is captured with a symbol for classroom discussion and the second question is captured with a symbol for partner work. Under the second question, participants see four people with speech bubbles: "2 pounds brown bread, please!", "I gained 8 pounds on vacation.", "We harvested 3 hundredweights of apples.", and "The baby weighs 6 1/2 pounds." (Becherer et al., 2010, p. 71; author's translation).

Attributes	Sofia Kowalewskaja	Pound & Hundredweight
challenging		X
not so challenging		
text-intensive	X	X
many tasks		
no reference to the present	X	X
reference to the present		

Table 1. Repertory Grid of Participant 31

Attributes	Sofia Kowalewskaja	Pound & Hundredweight
active calculating required		X
reading task	X	
complex		X
clear	X	
topicality reference		X
purely historical	X	

Table 2. Repertory Grid of Participant 55

The results of the presented pilot study are organized in a repertory grid, where each item/ task can be assigned to one or more attributes. Table 1 shows the repertory grid of participant 31. This participant named six attributes while he compared the presented historical snippets: challenging, not so challenging, text-intensive, many tasks, no reference to the present, and reference to the present. The participants compare three mathematical historical tasks; therefore, some attributes aren't assigned to the two presented tasks. Participant 31 perceives the task "Sofia Kowalewskaja" as text-intensive and with no reference to the present. The task "Pound and Hundredweight" seems to him challenging, text-intensive and with no reference to the present. The last attribute is surprising, since pounds and hundredweights are still in use in Low German language.

Table 2 shows the repertory grid from participant 55. This participant named six attributes: active calculating required, reading task, complex, clear, topicality reference, and purely historical. The participant's connotation of the task about

Sofia Kowalewskaja is “reading task”. This task seems to him clear and purely historical, while the task “Pound & Hundredweight” require active calculating, is complex, and has topicality reference. Surprisingly, Participant 55 doesn’t attribute “active calculating required” to the “Sofia Kowalewskaja” task, although there are some questions for calculation the years.

CONCLUSION

In this explorative survey, teachers’ beliefs are collected in a repertory grid within an online interview. All repertory grids from the 143 participants of the pilot study were analysed. The results show different views on historical snippets, such as “many tasks” and “clear” There are attributes related to the appearance of the tasks (text-intensive, pictures, clear, complex), to the content (eurocentric, purely historical), some mathematical remarks (unique solution, problem-solving, active calculating required) and related to use in the classroom (tasks directly address students, reading task). At the same time, there are some similarities, such as “topicality reference” and “reference to the present” or “text-intensive” and “reading task”. Particularly the similarities allow a comparison of the formulated benefits on history of mathematics in education. Reading literacy, like the participant associate, is not one of the benefits that researchers formulate for these tasks. This gap between teachers’ interpretation and researchers’ claims are possibly an indication for a misunderstood use of tasks about history of mathematics in classrooms.

Additional information

For a closer look into the online survey, use the following link (online available in German until ICMT3):

https://surveys.hrz.uni-giessen.de/limesurvey/index.php/794939?lang=de

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224. https://doi.org/10.1023/A:1009903206236.

Becherer, J., Gmeiner, H., Hübscher, H., Huschens, H. J., & Schulz, A. (2010). *Duden Mathematik 4*. Berlin, Germany: Duden.

Brown, M. W. (2009). The teacher–tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). New York, NY: Routledge.

Bruder, R., Lengnink, K., & Prediger, S. (2003). Wie denken Lehramtsstudierende über Mathematikaufgaben? Ein methodischer Ansatz zur Erfassung subjektiver Theorien mittels Repertory-Grid-Technik. *Mathematica Didactica, 26*(1), 63–85.

Bütüner, S. Ö. (2018). Secondary school mathematics teachers’ knowledge levels and use of history of mathematics. *Journal of Education and Training Studies, 6*(1), 9–20.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education, 45*(5), 633–646.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*, 391–408.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199–218.

Hersh, R. (1979). Some proposals for reviving the philosophy of mathematics. *Advances in Mathematics, 31*(1), 31–50.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*. Washington, DC: NCES.

Howson, G. (1995). *Mathematics Textbooks: A comparative study of grade 8 texts*. Vancouver, Canada: Pacific Educational Press.

Kelly, G. A. (1955). *The psychology of personal constructs*. New York, NY: Norton.

Liu, P. (2003). Do teachers' need to incorporate the history of mathematics in their teaching? *Mathematics Teacher, 96*(6), 416–421.

Remillard (2018). Examining teachers' interactions with curriculum resource to uncover pedagogical design capacity. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 69–88). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-73253-4_4

Schorcht, S. (2018a). History of mathematics in German mathematics textbooks: Typology of tasks. In K. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership* (pp. 143–162). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-73924-3

Schorcht, S. (2018b). *Typisierung mathematikhistorischer Beispiele in deutschen Mathematikschulbüchern der Klassenstufe 1 bis 7*. Münster, Germany: WTM-Verlag.

Tzanakis, C., & Arcavi, A. (2002). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 201–240). Dordrecht, Netherlands: Kluwer.

Wittmann, E. C., & Müller, G. N. (2011). *Das Zahlenbuch 3*. Stuttgart: Ernst Klett.

Xenofontos, C., & Papadopoulos, C. E. (2015). Opportunities of learning through the history of mathematics: The example of national textbooks in Cyprus and Greece. *International Journal for Mathematics Teaching and Learning*, 1–18. Retrieved from http://www.cimt.org.uk/journal/ xenofontos.pdf

RESEARCH ACTUALITY AND TREND ON THE MATHEMATICAL REPRESENTATIONS OF TEXTBOOKS

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To understand current state of the research on the mathematical representation of textbooks, 211 articles in total on mathematical representation are collected in ERIC and CNKI to explore the research objects, with the result of only 19 papers focusing on mathematics textbooks. Then, those papers are analysed in terms of research phases, research contents, research problems and research methods. Results show that 1.) those studies focused on elementary and junior high school 2.) research contents are mostly about arithmetic, algebra and geometry. 3.) there are four main research problems, and most of the studies presented or compared the characterization of the mathematics textbooks. 4.) 78.26% of the articles are used the document analysis method or comparative research method. The trend of the mathematical representation about textbooks are studied at last.

INTRODUCTION

Representation is an important concept of cognitive psychology. After being introduced into the field of mathematics education, representation is widely used to learn the meaning of mathematical concepts and mathematical problems that are quite abstract by nature (De Bock, Van Dooren, & Verschaffel, 2015). In addition, textbooks are a key component of the intended curriculum which have substantial influence on teachers' teaching and students' learning (Zhu & Fan, 2006). According to the TIMSS survey, it has shown that the majority of mathematics teachers used textbooks as the main written source that they selected teaching approaches (Beaton, 1996). Therefore, increasing researchers began to pay attention to the mathematical representation of textbooks. For example, Österholm investigated the language representation of the mathematical representation in textbooks (Österholm, 2013). Some studies indicated that representations of textbooks played a significant role in the process of teaching and learning (Fan & Zhu, 2000; Mcgee & Moore-Russo, 2015). For an in-depth understanding of mathematical representation in mathematics textbooks, in the present study, we investigate recent literatures that are related to the mathematical representation and attempt to study current state on the mathematical representations of textbooks and its trend as well. Then, some questions are raised as follows: What's the overall trend of the articles that focused on mathematics textbooks in the field of mathematical representation? What's the characteristics of research phases, contents, problems and methods among the studies regarding to the mathematical representations of mathematics textbooks?

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 305–310). Paderborn: Universitätsbibliothek Paderborn.

REESEARCH PROCESS

Research Method

Meta-analysis is first used to obtain quantitative data, which can be used to synthesize all the different research results in a field as much as possible and also can be analysed by statistical methods (Hart, Smith, Swars, & Smith, 2009). Then, content analysis method is adopted to explore the nature of the content which is a kind of repeatable and effective way and can make specific inferences from its text attributes or features (Bikner-Ahsbahs, Knipping, & Presmeg, 2015).

Dataset

The Education Resources Information Centre (ERIC) is one of the most popular search engines used by educational researchers and practitioners internationally. And China National Knowledge Infrastructure (CNKI) is one of the world's largest continuous dynamic update of the full-text databases for Chinese academic journals. Thus, it is comprehensive to use these two databases to search the international and domestic literatures related to mathematics education.

Firstly, skim the titles and abstracts of the papers which are collected with key words "mathematics education; mathematical representation" in two main bibliographic databases: ERIC and CNKI. Then, select articles or documents that are related to the mathematical representations and are downloadable with full-text. Three months later, search again to avoid the omission. Finally, totally 211 papers with regard to mathematical representation are selected, of which 156 articles are from 1973 to May of 2019 in ERIC and 55 articles are from 1994 to May of 2019 in CNKI. While only 19 articles are involved with mathematics textbooks (see Table 1).

The number of articles	About mathematical representation	About mathematical representation of textbooks
ERIC	156	15
CNKI	55	4
Amount	211	19

Table 1. The number of the articles

Coding Process

The coding process is as follows: First of all, encode all the 211 articles in order after skimming; Secondly, browse these articles again with the purpose of recording the corresponding authors, publications years, research objects and also marking the typical features of their references; Finally, read intensively the 19 articles that are involved with mathematics textbooks and analyze their research sections, research contents, research problems and research methods.

RESULTS AND PRELIMINARY CONCLUSITONS

Results of Overall Trend

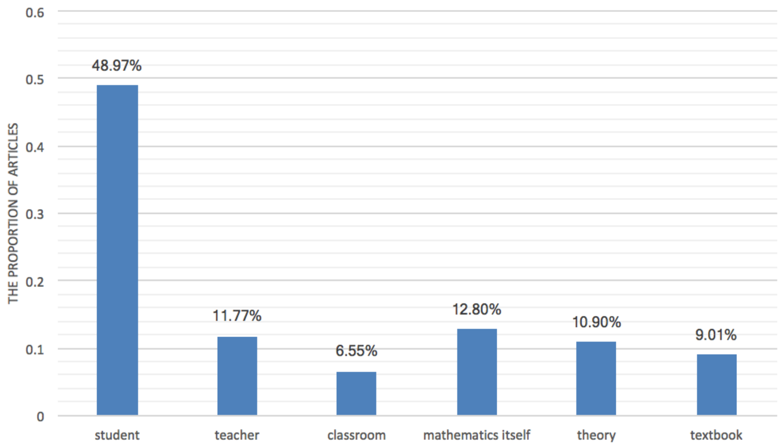

Figure 1. The proportion of different research objects

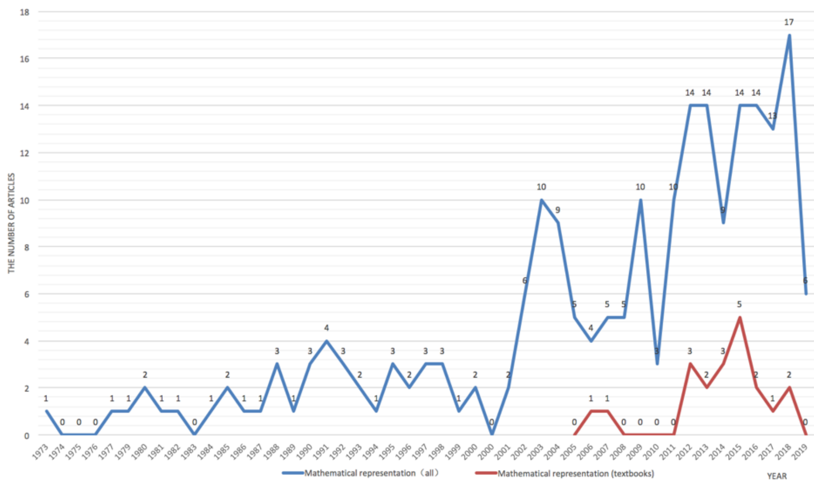

Figure 2. The number of articles in different years

As clearly shown in Table 1, only 15 out of 156 articles from ERIC focus on mathematical representation of textbooks and only 4 out of 55 articles from CNKI do so. Figure 1 displays the distribution of research objects among 211 articles. Result shows that there are six research objects when studying the mathematical representation, namely, student, teacher, classroom, mathematics itself (e.g., mathematical concepts representation), theory (e.g., theoretical discussion) and textbook. And it is found that most studies focused on the student' representational ability, while only 9.01% studies focus on the mathematical textbooks.

From the perspective of time point of view (Figure 2), few studies focused on the representation of textbooks although there were some studies on mathematical representation before 2000. After 2000, researches on mathematical representation were increased rapidly. Besides, researchers tended to pay much more attention to the textbook representation from after 2015.

Results of Mathematics Textbooks

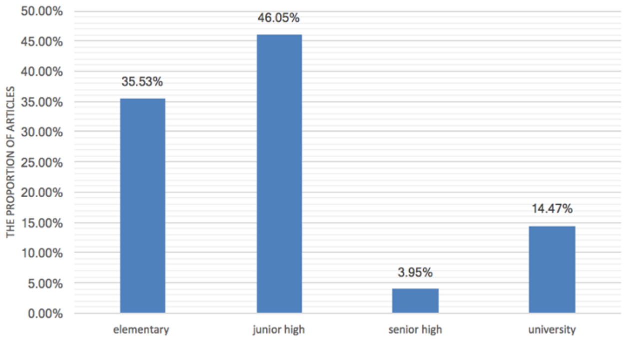

Figure 3. The proportion of different research phases

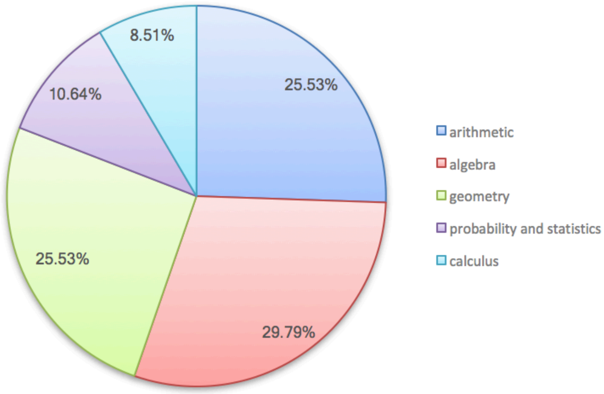

Figure 4. The proportion of research contents

The school sections are divided four parts except the stage of kindergarten. It can be seen that the research on representation of textbooks is mainly concentrated in elementary and junior high school but less on senior high school (see Figure 3). In addition, as revealed in Figure 4, it is easily found that studies pay more attention to the regular courses such as arithmetic (e.g., number and operation), algebra (e.g. function) and geometry (e.g., measurement). For instance, the coordination of mathematical representations is particularly important in the domain of functions (Nyikahadzoyi, 2015). However, only a few studies are involved with the probability and statistics, which are currently widely mentioned and applied for the population of artificial intelligence (AI).

The types of research problems	Amount	Examples
Describing the characteristics of a textbook	6	What is the prevalence of different CMR tasks in a reform calculus textbook? (Chang, Cromley, & Tran, 2016)
Comparing the representations in different versions of textbooks	9	What is the similarities and differences in the representation of problems in the selected textbooks? (Zhu & Fan, 2006)
Exploring the changes of representations in textbooks with the technology	2	Years from now will there be no paper mathematics textbooks? (Usiskin, 2018)
Exploring the connections of the students' representational ability with the use of the representations of mathematics textbooks by students and teachers.	2	To what extent are students encouraged or required to use representations to complete math tasks as a means of developing and consolidating their representational ability in sixth- and seventh- grade mathematics textbooks? (Garderen, Scheuermann, & Jackson, 2012)

Table 2. The types of research problems

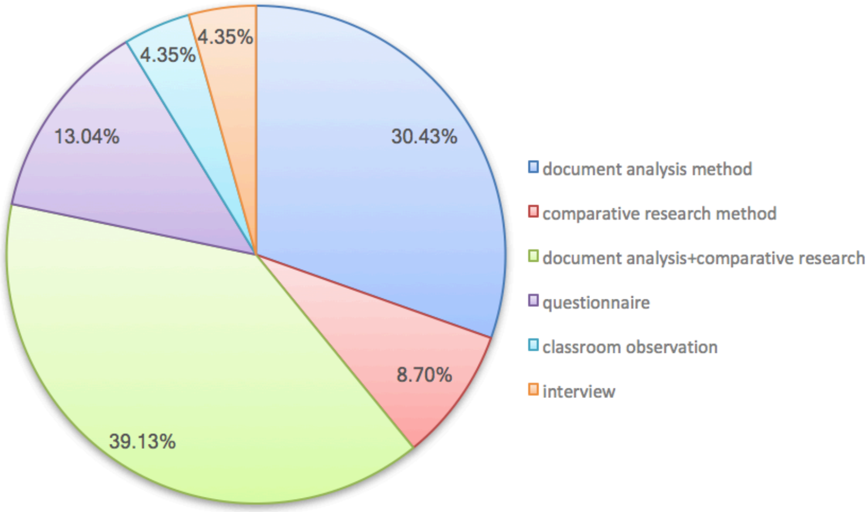

Figure 5. The proportion of different research methods

Among these 19 articles that are related to the textbook representation, there are four main research problems (see Table 2) and six methods (see Figure 5). Document analysis method and comparative research method are common in studying textbooks, so many researchers didn't distinguish the difference between them. Here, the document analysis is a way for studying documents by coding. And comparative research method refers to the study of two or more educational entities in order to investigate how and why they are alike or different. "Document analysis +comparative research" means that the study is used both methods. It is found that most of the studies presented or compared the characters of the mathematics textbooks using document analysis (a way for studying documents by coding) and comparative research methods (78.26% of articles have used these two methods). Therefore, only a few studies were involved with the technology or students' representational ability. And methods of interview, classroom observation, and questionnaire survey were rarely adopted.

PRELIMINARY CONCLUSIONS

Based on the above literature review, it's not difficult to find that current researches on the representation of mathematics textbooks are still in its infancy. The current research problems regarding to the representation of textbooks are relatively simplistic. In particular, few studies have connected with students, teachers or the technology. Besides, the research methods and tools are relatively simple. Although some studies have focused on the mathematics representation of textbooks, with the enrichment of e-textbooks and other forms of textbooks, more and more attention is suggested to be paid to the mathematical representation of textbooks. In the future, the research problems should be broad, such as to be involved with the transformation of the textbooks, the changes of the mathematical representation of textbooks with the improvement of technology and the connection with users. Also, research methods on the mathematics textbooks need to be more diversified.

References

Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1996). *Mathematics achievement in the middle school years: IEA's third international mathematics and science study*. Boston College: Chestnut Hill: TIMSS International Study Center.

Bikner-Ahsbahs, A., Knipping, C., & Presmeg, N. (2015). *Approaches to qualitative research in mathematics education*. Dordrecht: Springer.

Chang, B. L., Cromley, J. G., & Tran, N. (2016). Coordinating multiple representations in a reform calculus textbook. *International Journal of Science and Mathematics Education, 14*(8), 1475–1497. doi:10.1007/s10763-015-9652-3

De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine function: two studies on the role of external representations *International Journal of Science and Mathematics Education, 13*(1), 47–69.

Fan, L., & Zhu, Y. (2000). Problem solving in Singaporean secondary mathematics textbooks. *The Mathematics Educator, 5*(1/2), 117–141.

Garderen, D. V., Scheuermann, A., & Jackson, C. (2012). Developing representational ability in mathematics for students with learning disabilities: A content analysis of grades 6 and 7 textbooks. *Learning Disability Quarterly, 35*(1), 24–38.

Hart, L. C., Smith, S. Z., Swars, S. L., & Smith, M. E. (2009). An examination of research methods in mathematics education (1995–2005). *Journal of Mixed Methods Research, 3*(1), 26–41. doi:10.1177/1558689808325771

Mcgee, D. L., & Moore-Russo, D. (2015). Impact of explicit representation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. *International Journal of Science and Mathematics Education, 13*(2), 357–384.

Nyikahadzoyi, M. R. (2015). Teachers' knowledge of the concept of a function: A theoretical framework. *International Journal of Science and Mathematics Education, 13*(2), 261–283.

Österholm, M. (2013). What is so special about mathematical texts? Analyses of common claims in research literature and of properties of textbooks. *ZDM Mathematics Education, 45*(5), 751–763.

Usiskin, Z. (2018). Electronic vs. paper textbook presentations of the various aspects of mathematics. *ZDM Mathematics Education, 50*(5), 849–861. doi:10.1007/s11858-018-0936-2

Zhu, Y., & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from mainland China and the United States. *International Journal of Science and Mathematics Education, 4*(4), 609–626. doi:10.1007/s10763-006-9036-9

WHY THE TEXTBOOK MATTERS —
A TWO-DOMAIN IMPACT ANALYSIS

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In primary schools around the world, mathematics textbooks are the most relevant medium for lesson preparation for the vast majority of teachers. Though the field of textbook research is constantly growing, our knowledge about textbook effects on students learning is still rather constrained. In this contribution, three studies are presented, which address that research gap for the field of primary school arithmetic by using a large-scale longitudinal sample of 1664 students from Grade 1 to 3. The first study shows general effects of the textbooks on the students' mathematical achievement in arithmetic over the first three years of schooling. Studies 2 and 3 deepen this analysis by focusing on two domains of strategy competence, namely the use of arithmetic principles (Grade 1) and an adaptive strategy choice (Grade 3), and explain the effects by the quality of opportunities to learn presented in the textbooks.

THEORETICAL BACKGROUND

The mathematics textbooks are a key resource for teaching and learning mathematics in primary school: They represent and translate the abstract curriculum (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002) and are extensively used in everyday classroom practice (Mullis, Matrin, Foy, & Arora, 2012). Though they vary in both their content and pedagogical style (Pepin & Haggarty, 2001), there is still a lack of empirical evidence to support the reasonable assumption that these differences affect student achievement. Previous research shows inconsistent results (van Steenbrugge, Valcke, & Desoete, 2013) and is often limited to small sample sizes and cross-sectional designs (Fan, Zhu, & Miao, 2013). Moreover, many existing studies consider textbooks as representatives for different curricula and thus are in fact curriculum studies. Based on this, our first study aims to analyse effects of different textbooks representing the same curriculum on student achievement. The second and third study deepen the results of Study 1. They examine the quality of opportunities to learn from those textbooks regarding the use of arithmetic principles and adaptive strategy choice and analyse the effect of the textbook quality on student achievement in those domains. Both of them represent forms of strategy competence. These two domains were chosen because they are two important elements of the primary school curriculum but yet they are often mastered insufficiently by the students. One reason for the missing success might be a lack of quality of opportunities to learn in the mathematics classroom, which are affected by the textbook used.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 311–316). Paderborn: Universitätsbibliothek Paderborn.

Arithmetic principles

Students' development from the use of simple counting strategies towards the use of advanced calculation strategies – or even recall – to solve addition or subtraction problems up to 20 is a major goal of the first years of schooling (Kilpatrick, Swafford, & Findell, 2001). In this progress, children "do not move from knowing nothing about the sums and differences of numbers to having the basic number combinations memorized" (Kilpatrick et al., 2001, p. 182), but gradually develop more advanced and abstract solution methods for such problems by perceiving and utilizing the underlying arithmetic principles. However, counting strategies form a major approach for many students beyond Grade 1 (Benz, 2005). The persistent use of them is troublesome, since it causes problems like a higher error rate or difficulties to perceive relations between numbers or operations (Padberg & Benz, 2011). Four basic tasks generate the family of facts, the core of arithmetic principles in the first grade: A starting task $a + b = c$, its commutative equivalent $b + a = c$, and the two inversions $c – a = b$, $c – b = a$. For each task four neighbour tasks arise by decreasing or increasing one addend, the subtrahend or the minuend by 1 (e.g., $(a + 1) + b = (c + 1)$). We refer to the family of facts and the neighbour tasks as basic arithmetic principles. In addition, further strategies like decomposing, simplifying or doubling belong to the arithmetic principles of Grade 1.

Adaptive strategy choice in multi-digit addition and subtraction

In the last two decades a broad consensus about the importance of an adaptive strategy use evolved (e.g., Verschaffel, Greer, & De Corte, 2007). Adaptivity in this case is described as "the ability to creatively develop or to flexibly select and use an appropriate solution strategy in a (un)conscious way on a given mathematical item or problem, for a given individual, in a given sociocultural context" (Selter, 2009, p. 624). For multi-digit addition and subtraction up to 1000, we distinguish between digit-based strategies, namely the written algorithms, and number-based strategies, which can be executed both mentally and in written form. While a digit-based algorithm constitutes a uniform approach to all kinds of problems, number-based strategies offer a variety of solution approaches. The most common number-based strategies are presented in Table 1. Adaptivity in this context is indicated by an adaptive use of strategies to find efficient solutions to given arithmetic problems. While first longitudinal studies revealed that an adaptive strategy use is accompanied by a broader conceptual knowledge or a deeper understanding of base-ten number conceptions (e.g., Fuson et al., 1997), empirical studies repeatedly reported a lack of adaptivity in students' actual strategy use (e.g., Csikos, 2016; Heinze, Marschick, & Lipowsky, 2009; Torbeyns & Verschaffel, 2016).

The development of strategy competence

Siegler divides the development of strategy competence into four dimensions, of which he could show that each of them contributes to the primary competence (cf. Lemaire & Siegler, 1995): *strategy repertoire*, the knowledge of different types of strategies;

strategy distribution, the knowledge of relative frequencies these strategies are used; *strategy efficiency*, the ability to perform strategies quickly and accurately; and *strategy selection*, the ability to flexibly select a strategy on a given problem.

Stepwise	Split	Compensation	Simplifying	Indirect Addition
123 + 456 = 579	123 + 456 = 579	527 + 398 = 925	527 + 398 = 925	701 - 698 = 3
123 + 400 = 523	100 + 400 = 500	527 + 400 = 927	525 + 400 = 925	698 + 3 = 701
523 + 50 = 573	20 + 50 = 70	927 – 2 = 925		
573 + 6 = 579	3 + 6 = 9			

Table 1. Common number-based strategies for multi-digit addition and subtraction

Present studies

The present studies aim to contribute to current textbook research, investigating textbook effects on students learning by reanalysing a large-scale longitudinal sample of 1664 students from 93 classes from Grade 1 to 3. Due to sample mortality the sample sizes reduces to 1579 in Study 2 and 1404 in Study 3. The data comprise teacher and student information including their arithmetic skills at the end of Grade 1 and 2, solutions on arithmetic principle tasks at the end of Grade 1 and on multi-digit addition and subtraction problems at the end of Grade 3, as well as their scores of a nationwide mathematics competence test at the end of Grade 3. The sample was selected in one federal state of Germany, thus all textbooks used follow the same curriculum. The classes in this sample used one of four common textbooks which were distributed relatively evenly across the classes. Hence, this sample allows to examine the effects of textbooks representing the same curriculum on students' mathematics achievement with a longitudinal design and a sound sample size. Accordingly, our studies address the following research questions:

1. Does the textbook choice of primary mathematics teachers have an effect on the development of students' mathematics achievement in Grade 1-3?
2. Does the mathematics textbooks' quality concerning basic arithmetic principles affect the students' ability to make use of them at the end of Grade 1?
3. Does the mathematics textbooks' quality concerning adaptive expertise in multi-digit addition and subtraction affect the development of primary school students' adaptive expertise in this field at the end of Grade 3?

METHOD

In order to rate the textbook quality regarding the use of arithmetic principles and adaptive expertise, we derived specific criteria from the four dimensions of the Siegler model for both domains, respectively, and applied them to the textbooks of Grade 1 (arithmetic principles) and Grade 2 & 3 (adaptive strategy choice). Both ratings were carried out by three independent and trained persons (Fleiss' κ: .61 – 1.00). Students' learning progress at the end of Grade 1 and 2 was measured with curriculum based arithmetic tests (EAP/PV reliability = .93 and .94). At the end of Grade 3, the total scores from the national mathematics test for the two content areas “numbers” and “patterns” were provided by the schools. The ability to use arithmetic principles was assessed by six items of the end of Grade 1 test (WLE reliability: .82), the adaptivity of strategy choice by four items of the end of Grade 3 test (Cronbach’s $\alpha = .71$). In order to control for individual differences between students or classes and thus to exclude other possible influences, students’ learning prerequisites related to basic numerical skills, basic cognitive abilities, and German language skills were measured at school entrance using approved standardized tests. Teacher qualification and beliefs were collected by a questionnaire.

For examining effects of textbooks and their quality we conducted multilevel analyses which take into account the nested structure of the sample (students in classes). We included the variables for learning prerequisites at school entrance on individual level and the aggregated values (as an indicator of class composition) as well as the teacher variables on class level. In Study 1 the textbooks were included as dummy coded variables, in Studies 2 and 3 we included the textbook quality ranking scales for the respective domains. Missing data were handled by the Full Information Maximum Likelihood method (FIML).

RESULTS

Study 1 revealed significant differences in the students’ arithmetic achievement for all Grade levels. The explained variance on class level increase substantially when including the textbooks (ΔR^2 = 11.7–23.3 %). However, the textbooks differ in these effects. While at the end of Grade 1 only one textbook has a significant positive effect in comparison to the reference textbook, at the end of Grade 2 all three textbooks show positive significant effects, with varying effect sizes, and two of the textbook series also show such effects regarding the scores of the national mathematics test at the end of Grade 3. The regression coefficients vary between $\beta = .20$ and $\beta = .48$, which can be interpreted as effect size similar to Cohen’s *d* due to standardized scales. Thus, the textbook choice has an effect on the students’ development, and the effect size differs substantially between the textbooks as well as between the measurements after one, two, and three years of schooling.

The domain-specific analyses of the textbooks show crucial differences in both quality and quantity of opportunities to learn the textbooks present, with regard to arithmetic principles as well as to an adaptive strategy use. In both domains, there is one series with a notably lower instructional quality than the other three, though it is not the same series for both.

The analyses of Study 2 show substantial effects of the Grade 1 textbook quality concerning opportunities to learn for the use of arithmetic principles on the students’ actual use of these principles at the end of the first year of schooling. Students who used a textbook with a higher quality showed a significantly higher ability to use the arithmetic principles, with effect sizes up to .40 standard deviations.

Also Study 3 shows a significant effect of textbook quality on student achievement. In this study we examined the quality of opportunities to learn regarding an adaptive use of strategies for multi-digit addition and subtraction, presented in the textbooks of Grade 2 and 3. The results reveal a substantial effect of textbook quality of Grade 2 on the students' actual strategy use at the end of Grade 3. The effects size ranges up to .34 standard deviations. The textbook quality of Grade 3 shows no additional effect, which confirms former research results on the importance of an early emphasis of strategy use (e.g., Blöte, van der Burg, & Klein, 2001). Further, the analyses show an interaction of textbook quality and students' prior knowledge, indicating a particular benefit of a good textbook quality for high achieving students.

DISCUSSION

The three studies presented contribute to the existing research of textbook effects by analyzing the effects of four different primary school textbooks representing the same curriculum on student achievement with a longitudinal design and a sound sample size. The findings provide empirical evidence for the influence of textbooks on student achievement, and can explain this influence by the quality of opportunities to learn presented by the textbooks for two domains of arithmetic education.

If textbooks as a curriculum resources differ in their instructional quality, and these quality differences affect the students' learning, it should be our aim to supply students with textbooks best for their learning and to avoid the use of disadvantageous textbooks. Hence, our results make the case for a textbook approval, which presence differs among as well as within countries. The criteria should be theory-based and empirically proven. Consequently, more studies on the effects of textbook quality are needed to provide these criteria. In addition, our results suggest that teacher, who use the textbooks for their lesson preparation, should be trained how to use textbooks, so they are able to reflect on the quality of textbooks' opportunities to learn and, if necessary, compensate inadequate representations of the curriculum.

Secondary analyses cause certain limitations. Beneath a non-experimental setting, specific information of interest for our research is missing, like fine-grained data on the implementation of the teaching content or on the teacher knowledge. Despite of these limitations the data set has the advantage that it covers a large sample taught by the same curriculum and allows multilevel analysis with an adequate explanatory power. Furthermore, we were able to assess and examine the effects of textbooks and their quality on student achievement for two specific domains representing strategy competence. Accordingly, we were able to supplement and further develop existing research on the effects of textbooks on students' learning.

References

Benz, C. (2005). *Erfolgsquoten, Rechenmethoden, Lösungswege und Fehler von Schülerinnen und Schülern bei Aufgaben zur Addition und Subtraktion im Zahlenraum bis 100*. Hildesheim: Franzbecker.

Blöte, A. W., van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology, 93*, 627–638.

Csíkos, C. (2016). Strategies and performance in elementary students' three-digit mental addition. *Educational Studies in Mathematics, 91*(1), 123–139.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education, 45*(5), 633–646. https://doi.org/10.1007/s11858-013-0539-x

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I. Carpenter, T.P., Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers. Adaptive strategy use and the influence of instruction in German third grade. *ZDM Mathematics Education, 41*(5), 591–604. http://doi.org/10.1007/s11858-009-0205-5

Kilpatrick, J., Swafford, J., Findell, B. (2001). *Adding It Up*. Washington, D.C.: National Academies Press. https://doi.org/10.17226/9822

Lemaire, P., & Siegler, R. S. (1995). Four aspects of strategic change. contributions to children's learning of multiplication. *Journal of Experimental Psychology: General, 124*(1), 83–96.

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics.* Chestnut Hill, Mass.: TIMSS & PIRLS International Study Center.

Padberg, F., & Benz, C. (2011). *Didaktik der Arithmetik* [Arithmetic Education]. Heidelberg: Spektrum Akademischer Verlag.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *ZDM Mathematics Education, 33*(5), 158–175.

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM Mathematics Education, 41*(5)*,* 619–625.

Torbeyns, J., & Verschaffel, L. (2016). Mental computation or standard algorithm? Children's strategy choices on multi-digit subtractions. *European Journal of Psychology of Education, 31*(2), 99–116.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). According to the book. Using TIMSS to investigate the translation of policy into practice through the world of textbooks. Dordrecht: Kluwer.

van Steenbrugge, H., Valcke, M., & Desoete, A. (2013). Teachers views of mathematics textbook series in Flanders. Does it (not) matter which mathematics textbook series schools choose? *Journal of Curriculum Studies, 45*(3), 322–353.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 557–628). (2nd edition). New York: MacMillan.

INADEQUATE LEARNING SEQUENCE AND ERRONEOUS FACT-LIKE STATEMENT IN A MATHEMATICS TEXTBOOK: WHAT CAN STUDENTS TAKE FROM THEM?

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To be used in Mexican public schools, all mathematics textbooks have to pass an allegedly rigorous review process by Ministry of Public Education (Secretaría de Educación Pública). Nevertheless, various revisions of approved mathematics textbooks reveal many deficient features, going from orthographical, mathematical and contextual errors to inadequate learning sequences. The aim of this initial exploration study was to find out what can secondary-school students (N = 143) take from an inadequate learning sequence and an erroneous fact-like statement in the mathematics textbook they use. The inadequate learning sequence was related to positive and negative numbers. Its inadequacy comes from (a) artificial context; (b) unclear questions regarding concepts and calculations; and (c) unnecessary complicated drawing tasks. Erroneous fact-like statement, related to the age of young persons should have in order to work legally, contains the affirmation that "four of ten" is 37 %. The results show that students' performances are influenced negatively by unclear questions and unnecessary complication of the drawings tasks. Although many students were able to detect the erroneous percentage value in the statement, they differ greatly in their argumentative skills. The students with poor skills only say "something is wrong with the percentage", while those with good skills affirm "the percentage is erroneous because "four in ten" is not 37 % but 40 %".

INTRODUCTION

Investigations on mathematics textbooks, ranging from content analysis to their use by teachers and students, become an important part of mathematics education research. It is evidenced recently by two international conferences, carried out in 2014 (United Kingdom) and 2017 (Brazil) and dedicated to various topics related to mathematics textbooks. Although the field still doesn't have a specialized research journal, there were a few review articles (Fan, Miao, & Zhu, 2013; Schubring & Fan, 2018) and books (Fan et al., 2018) giving the state of art.

Since long time, research in mathematics education paid attention to students' errors in learning school mathematics (Radatz, 1980; Newman, 1983; Zaslavsky, 1987). Much less attention was paid to the errors made by the authors of mathematics textbooks, mainly related to usage of invented and unreal "problem contexts" that created an inadequate image of "mathematics applications" in education (Pollak, 1968; Korsunsky, 2002).

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RESEARCH ON MATHEMATICS TEXTBOOKS IN MEXICO

In Mexico, there aren't too many articles and thesis reporting research results on mathematics textbooks. Among them two approches can be noticed. In the first approach, researchers are focused on different aspects of textbooks' contents.

Cordero, Cen, and Suárez (2010) were interested to analyze, using framework based on socio-epistemological theory, the uses of graphs that generate institutional practices in high school. They argue that the functionings and forms of graphs maintain a dialectical relationship, which expresses the development of the use of the graph in three aspects: the methods for the use of graphic representation, the understandings of graphs and their functionality.

Quiroz and Rodríguez (2015) analyzed the mathematical modeling praxeology of the mathematics textbooks for primary school students in Mexico and presented a detailed description of the types of mathematical modeling tasks given in the lessons of the books. Their results give an idea of the mathematical modeling cycle through its praxeological elements. They concluded that the textbooks that the Secretary of Education provide to students are in a poor agreement with the objectives proposed in the actual curriculum.

Castañeda, González, and Mendo-Ostos (2017) analyzed and clasifed the type of mathematical problems and problem-solving strategies in a sample made up of the five most widely-used mathematics textbooks for the first year of junior high school in the 2013– 2014 school year in Mexico. They concluded that the textbooks focus on providing a comprehensive range of problems that, for the most part, do not allow multiple problem-solving strategies, whereas others can be solved using an algorithm or direct procedure.

In the second approach, developed in the Master program in Mathematics Education at the Benemérita Universidad Autónoma de Puebla, researchers are interested to explore what students (Monterrosas et al., 2018) or teachers (Ruiz, Slisko, & Nieto, 2018) do when asked to interact with a selected (defective) problem of a textbook.

A more focused investigation on students dealing with textbook problems on balance model (Zamora Corona, 2017) provided initial evidence that schooling time and problem features significantly influenced students' performances. Secondary (N=102) and high-school students (N=110) were asked to solve two problems. One problem had a solution and the other, with contradictory data, was insolvable. Secondary-school students performed rather poorly. Only 11 students solved the first problem and only one student was able to detect that second problem had contradictory data. High-school students were better: 55 students found the solution of the first problem and 15 stated that the second problem was not formulated properly.

AIM AND METHODOLOGY OF THIS RESEARCH

In the research resumed above, two problems were taken from mathematics textbooks that students didn't use and that didn´t correspond to the grades they were studying. In fact, the textbooks were written for two or more grades below their grade.

In this research we wanted to explore:

1) How do students perform on a problem with an inadequate sequence from their textbook that they should have solved before?
2) Are students able to detect a trivial mathematical error in a fact-like statement in their textbook they didn't deal before?

In the first phase, the content of the mathematics textbook in question was carefully analysed. Following the practice of Pollak and Korsunsky, the names of the authors and the publishing company won't be revealed. Many linguistic and calculation errors were detected. Some might say that these errors are due to careless revision processes in the publishing company and Ministry of Education. Nevertheless, there are more serious errors that could not be "defended" in that way. As examples come two errors:

> **Example 1.** "To construct (draw) a triangle, the length of one of its sides must always be less than the sum of the other two."

This erroneous "rule" might lead students to conclude in a posterior "application task" that triangles with sides (3, 4, 7) and (3, 3, 8) are possible.

> **Example 2.** "In a meteorological station a balloon is launched into the stratosphere at 4 km from sea level to study the temperature variation.
>
> a) On the esplanade of the laboratory the first record is made, which turns out to be 36 °C; at 10 km a decrease of 72 °C is recorded. What is the temperature at that altitude?
>
> b) At 18 km a temperature of -72 °C is recorded. How many degrees decreased from the previous reading to 10 km?"

The troposphere, between 10 km and 20 km of height, has a constant temperature of -56 °C, so the data provided in the problem formulation are incorrect.

To explore students' performance, two parts of the textbook were selected. The first task was from the textbook part that students already knew, and the second task was taken from the part still unknown to the students.

> **Task 1**
>
> Felipe is in charge of a mine. In his reports he indicates the meters below ground level, using the notation of negative numbers. For example, -5 m indicates a depth of 5 m below ground level.

Figure 1. (Torre de vigilancia – The watchtower; Primera capa de piedra sedimentaria – The first layer of sedimentary stone)

1.1. The watchtower of the mine is 10 m above the ground. The first layer of sedimentary stone is 10 m below the ground. Explain the difference between these two values.

1.2. Draw on the illustration an antenna with a height of 20 m above ground level and a dinosaur skeleton located 20 m below the ground.

1.3. How many meters are there between the top of the antenna and the skeleton?

1.4. Felipe wrote in his exploration log: "We have reached -45 m, we dig a horizontal line of 5 m, then we dig down 7 m." How deep is the new exploration line?

This learning sequence is inadequate because of (a) its artificial, non-authentic context; (b) unclear questions regarding concepts and calculations; and (c) unnecessary complicated drawing tasks. None of these features help students really learn involved concepts and operations.

Task 2.

In a math textbook, the following text appears: "In Mexico, by law, the minimum age for admission to employment is 15 years. Before that age, the right of children to education governs. However, according to the Inegi, four out of ten (37%) children under 5 to 17 years of age do not attend school." What mathematical error have the authors of that textbook made?

These two paper-and-pencil tasks were given to 143 first-year secondary school students (corresponding to seventh grade in other countries) who had 50 minutes to finish them individually.

RESULTS AND CONCLUSIONS

Taking into account that the subtask 1.1. has an incorrect formulation (both given values are just "10 m"), students' performance was rather good: 78 students (almost 55%) were able to "explain" the difference by connecting "+10 m" with the height of the tower and "-10 m" with the depth of the sedimentary layer. This result can be understood as a natural consequence of their previous experience with this task in which they had teachers' assistance. Nevertheless, the rest of students interpreted the word "difference" as a "distance" (of 20 m) or were totally confused.

In the subtasks 1.2 (drawing of the antenna and the skeleton) and 1.3 (distance between the top of the antenna and the skeleton), the students' performances were poor. Only 7 students were able to provide a correct drawing (the antenna and skeleton with right dimension on the same vertical line) and a correct distance (40 m). 81 students gave a correct (but likely memorized) answer for the distance but their drawings were incorrect. The rest of the students had erroneous distances and drawings.

In the subtask 1.4, 31 students were able to find the correct answer (-52 m), while 17 students were misled by the question (how deep?) and gave, strictly speaking, an incorrect answer (52 m). 40 students used in their calculation all given numbers getting wrong answers (57 m or -57 m). 55 students were "lost" and unable to provide a clear reasoning behind their arbitrary "solutions".

In the task 2, 69 students were able to detect the error in the percentage, but only 18 of them provided an argument for their evaluation (four out of ten isn't 37 % but 40%). The rest of the students couldn't spot sought mathematical error but mentioned other confusing features of the textbook statement or even an incredible linguistic error ("Inegi" is wrong, it should be "INEGI").

The results of this small-scale pilot study show that inadequate textbook learning sequences influence negatively students' performances, both at procedural level and coherence between visual and numerical representations. Namely, not a single student justified the distance on number line in the subtask as "40 m = 20 m – (-20 m)" and only a few of them provided correct (although unnecessary complicated) drawings of the antenna and the skeleton. If revision processes at the publishing companies and Ministry of Education are unable to eliminate inadequate learning sequences in Mexican mathematics textbooks, the teachers should help students avoid their negative influences. In some cases, students should be given tasks, similar to task 2, to detect and correct erroneous information authors use problem formulations. These tasks would give students an important opportunity to practice their careful reading and critical thinking and to learn that mathematics textbooks aren't error-free learning materials.

References

Castañeda, A., González Rodríguez, J. C., & Mendo-Ostos, L. (2017). Libros de Matemáticas para primer grado de secundaria en México: Problemas y estrategias de solución. *Revista Electrónica de Investigación Educativa*, *19*(4), 97–111.

Cordero Osorio, F., Cen Che, C., & Suárez Téllez, L. (2010). Los funcionamientos y formas de las gráficas en los libros de texto: Una práctica institucional en el bachillerato. *Revista Latinoamericana de Investigación en Matemática Educativa*, *13*(2), 187–214.

Estrada-Ruiz, H., Slisko, J., & Nieto-Frausto, J. (2018). Detección de errores y contradicciones en un problema de un libro de texto de matemáticas: Una exploración inicial del pensamiento crítico de los maestros. *Acta Latinoamericana de Matemática Educativa, 31(1)*, 106–114

Fan, L., Miao, Z., & Zhu, Y. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education*, *45*(5), 633–646.

Fan, L., Trouche, L., Qi, C., Rezat, S., & Visnovska, J. (Eds.). (2018). Research on mathematics textbooks and teachers' resources: Advances and issues. New York: Springer

Korsunsky, B. (2002). Improper Use of physics-related context in high school mathematics problems: Implications for learning and teaching. *School Science and Mathematics*, *102*(3), 107–113.

Monterrosas Castillo, Y., Ruiz Estrada, H., Slisko, J., & Fuchs Gómez, O. L. (2018). Soluciones estudiantiles de un problema de movimiento en un libro de texto de geometría analítica: Influencias del razoanimiento lógico y de reflexión cognitiva. *Latin American Journal of Physics Education, 12*(2), 2303-1–2303-11.

Newman, M. A. (1983). *Strategies for diagnosis and remediation.* Sydney: Harcourt, Brace Jovanovich.

Pollak, H. O. (1968). On some of the problems of teaching applications of mathematics. *Educational Studies in Mathematics*, *1*(1), 24–30.

Quiroz Rivera, S., & Rodríguez Gallegos, R. (2015). Análisis de praxeologías de modelación matemática en libros de texto de educación primaria. *Educación Matemática*, *27*(3), 45–79.

Radatz, H. (1980). Students' errors in the mathematical learning process: A survey. *For the Learning of Mathematics*, *1*(1), 16–20.

Schubring, G., & Fan, L. (2018). Recent advances in mathematics textbook research and development: An overview. *ZDM Mathematics Education*, *50*(5), 765–771.

Zamora Corona, Y. (2017). *El uso de la balanza para el aprendizaje de las ecuciones de primer grado: El caso de los libros de texto autorizados.* Unpublished master thesis. Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla.

Zaslavsky, O. (1987). An empirical classification model for errors in high school mathematics. *Journal for Research in Mathematics Education*, *18*(1), 3–14.

RESOURCES FOR TEACHING GRAPH THEORY FOR ENGINEERS — ISSUE OF CONNECTIVITY

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This paper deals with the issue of resources for teaching graph theory in the context of engineering education. For engineers, making connections between mathematical contents and the real world is fundamental to make sense of the mathematics learned. We use here the concept of connectivity to analyze a chapter of a graph theory course in French Engineering School. We highlight kinds of connections between different concepts, different registers and different topic areas that allow the teaching of graph theory. Finally, we present a discussion of findings and their implications.

INTRODUCTION AND CONTEXT OF STUDY

The present paper is a part of a PhD work where we aim to study, through the lens of the interactions with resources, the impact of the research activity on teaching practices at university. Based on previous research, our objective, in this article, is to provide an analysis of resources for teaching mathematics for non-mathematics study programmes (engineering education here). The study belongs to the growing body of research related to resources for teaching mathematics at the university (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). We choose graph theory, a branch of discrete mathematics, as an object of study. Many reasons underlie our choice: graph theory belongs to contemporary mathematics, it has grown over the past fifty years and plays a fundamental role in modern applied mathematics (Gross, Yellen, & Zhang, 2004). The points of view of some mathematicians, stated in the introduction of their discrete mathematics and graph theory textbooks, have also motivated our choice. For instance, West (2001) highlights that graph theory is a contemporary mathematical field and no consensus has been made on the choice of topics to include in a university course; he adds that graphs are used in the design of communication networks, and have wide applications in the areas of the computing, social and natural sciences, engineering, etc. However, we note that the institutional context could influence the choices and the process of designing resources in terms of many factors (academic path, profiles of students, etc.).

We are interested in particular in engineering education where the relation between theory and practice is crucial to make mathematics more relevant to students (Flegg, Mallet, & Lupton, 2012). According to Gueudet and Quéré (2018), engineers consider making connections between mathematical contents and the real world fundamental to make sense of the mathematics learned at university.

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Among eight textbooks we consulted for our PhD work, two are explicitly addressed to future engineers as part of their public: Discrete mathematics with applications (Epp, 2010); Discrete mathematics and its applications (Rosen, 2012). Rosen (2012) and Epp (2010) consider making connections between theory and applications essential in the learning of discrete mathematics. We aim to study the potential of making connections between graph theory and its application in a course designed for teaching in the academic path of engineering. Therefore, we seek to answer the following questions: Which connections between graph theory concepts appear in engineering courses? Which connections between graph theory and its applications appear in engineering courses?

To address these questions, we will try to clarify further in the paper what do we mean by "connections". It will take a specific sense with the concept of connectivity.

GRAPH THEORY – EULERIAN AND HAMILTONIAN PATHS

In this paper, we consider particularly contents related to Eulerian and Hamiltonian paths. These contents have a historical importance; it is claimed that graph theory originated with Euler's work on the Königsberg bridges problem, which led to the concept of the Eulerian graph (Gross, Yellen, & Zhang, 2004). They appear, in the first sections, as basics in graph theory in most of the consulted textbooks.

An Eulerian cycle is a cycle that traverses a graph passing through all edges exactly one time (Epp, 2010; Rosen, 2012). Such a cycle exists if all vertices of a connected graph have even degrees. A graph is Eulerian if it has an Eulerian cycle. This definition of an Eulerian cycle mobilizes the concepts of "graph", "connectedness" and "degree of a vertex". Other definitions of an Eulerian cycle mobilize concepts such as connected component, simple cycle, etc. Epp (2010) and Rosen (2012) emphasize on algorithmic proofs of the existence of an Eulerian cycle, but mostly on the construction of an Eulerian cycle using an algorithm rather than proving its mere existence.

Hamiltonian cycles traverse each and every vertex of a graph exactly once. In Epp (2010) and Rosen (2012), they are introduced in the same chapter as Eulerian cycles, and mobilize the concepts of connectedness and degree of a vertex. No conditions exist until now that are necessary and sufficient for the existence of such cycles. Epp (2010) and Rosen (2012) make connections between Eulerian and Hamiltonian graphs and other mathematical topic areas, other disciplines and real life. Eulerian and Hamiltonian graphs offer many possibilities of applications to real life examples that can contribute to engineering education. Therefore, they may offer a wide spectrum of choices for teachers to design a course in the context of engineering education.

MAKING CONNECTIONS IN THE TEACHING OF GRAPH THEORY

In the context of engineering education, "connections" can be related with (at least) three issues. The first is the network of connections between, and across, the available resources and the users of these resources. It is closely linked to the modification of learning processes in digital era that prompted Siemens (2005) to introduce the concept of *connectivism*. The second issue is the need to build links between mathematics courses and engineering courses. The third one is, within mathematics, the networking between mathematical objects and their representations (Hiebert & Carpenter, 1992). The latter is recognised as essential for the understanding of mathematical ideas and facts. Along the same lines, for Duval (2006), the acquisition of a mathematical object necessarily passes through the acquisition of one or several semiotic representations. According to Duval (2006), the mathematical concepts can be described through different register of semiotic representation (natural language, formalism, graph, etc.). A register of semiotic representation should allow the following activities: representing a mathematical object; treating representations within the register; converting representations from a register to another.

Based on these works, we consider the concept of *connectivity* (Gueudet, Pepin, Restrepo, Sabra, & Trouche, 2018) developed in the frame of e-textbooks analysis. This connectivity has two components: “macro-level connectivity” that refers to the potential of linking to and between users and resources outside the textbook (practical aspect); and “micro-level connectivity”, which focuses on connections in, between, and across individuals’ cognitive/learning tasks and activities, in a mathematical topic within the e-textbook. We therefore define connectivity in a set of resources for teaching graph theory as the connecting potential, practically and cognitively, for a given user (student or teacher). We only use, for this article, micro-level connectivity with the following criteria: connections among/between concepts, connections between different mathematical topic areas (particularly algorithmic), connections between disciplines, and connection between different semiotic representations.

The criteria of micro-connectivity we consider here could allow us to determine, in the case of graph theory, the connections established in resources between theory and application. We note here that we distinguish between “theory” and “application” from the point of view of the resources designers.

CASE OF RESOURCES FOR TEACHING AND METHOD OF ANALYSIS

We consider resources related to a course of introduction of graph theory in a French Engineering School. The course is part of the engineering specialisation “supply-chain management”. The related resources are collectively designed and used by teachers from different universities whose field of research is “operational research and combinatorial optimization”. They are available to the students on a Moodle platform and consist of: Java and Python files (introductory activities), a file entitled “graphs: algorithms and modelling” which covers all contents of the course, “Moodle areas” corresponding to 8 units of contents (slideshows, tutorials, etc.). We consider, for the analysis, the unit entitled “cheminements” (paths) (overviewed in the figure 1 below).

The platform displays three categories of objectives of the unit: (1) “techniques of proofs” in particular “recognising algorithmic proofs and applying them in simple problems”; (2) “properties” where focus is on characterising the minimal number of edges of a connected graph and Euler's theorem; (3) "paths" that consist in "recognizing and writing Breadth-First and Depth-First Search algorithms".

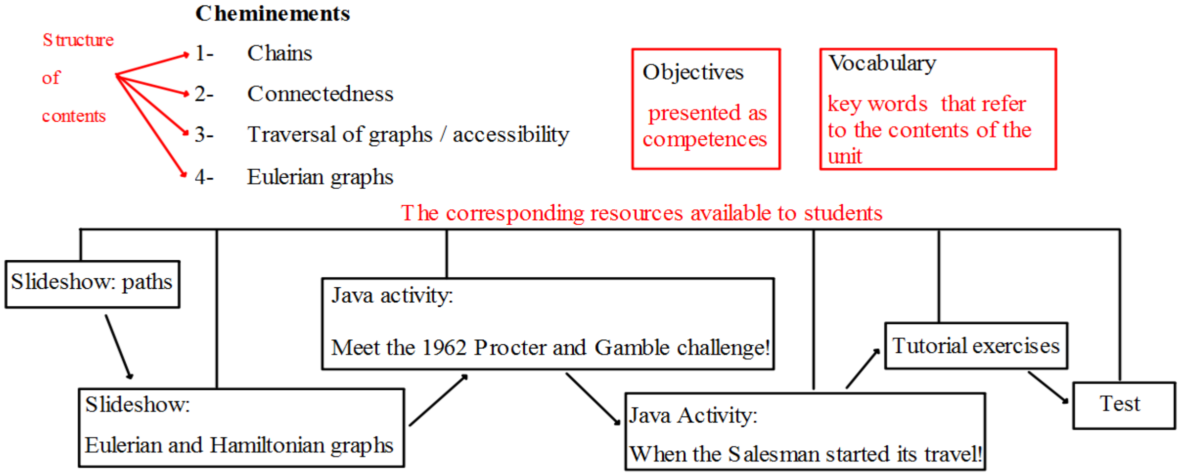

Figure 1. The description of the contents of the unit "cheminements"

To analyse the resources, in terms of micro-level connectivity, we use the grid developed in (Gueudet et al., 2018). We consider from the grid the elements that correspond to the criteria we retained (see above). Therefore, we analyse resources in terms of connections with/between: previous and further knowledge, concepts at stake, moments of appropriation of a concept (or connected concepts), mathematical topic areas, other disciplines, semiotic representations, real life problems and examples, software, strategies of solving exercises and variations of the same exercise. Our method of analysis consists of five steps: 1) we study the "progression" of concepts in terms of connections with previous and further knowledge; 2) we identify concepts used in the resources retained, focusing on the way they are connected by definitions, propositions, theorems, proofs, examples, and/or exercises; 3) for each concept (or connected concepts), we characterize the connection of different moments of appropriating them (for instance, progressive deepening of the concepts in the different parts of the course; or complete presentation of them in an independent part) and the context of this appropriation (different mathematical topic areas, different disciplines, real life situations, etc.); 4) we identify the registers of semiotic representation used for each concept, or that can be used in the case of questions or exercises, and the cognitive activities involved (represent, treat and convert) (Duval, 2006); 5) we determine the different strategies to solve exercises. In the fourth step, we identify, where appropriate, the connections made with particular software.

The five steps above allow us to understand the connections that teachers emphasize in the resources. Analysing the results from the lens of their objectives will enable us to determine the contribution of the connections made in reaching the objectives, and the alignment of the objectives with the context of engineering education.

ANALYSIS AND DISCUSSION

"Eulerian and Hamiltonian paths" involve many concepts of graph theory that can refer to "mathematical objects" such as cycles, length, connected graphs, etc. and concepts that refer to processes such as accessibility, breadth first and depth first traversals. We notice an emphasis on the connection between concepts that refer to processes and concepts that refer to mathematical objects such as Eulerian graph and depth first traversal, Hamiltonian graphs and depth first traversal, cycles and accessibility, etc. We will present here, following the five steps of resources analysis, the results found out in the case of the connected concepts: "Eulerian graph" and "depth first traversal".

A connection between the two mentioned concepts is introduced for the first time in the file "graphs: algorithms and modelling" in the problem of the "Chinese Postman". The problem aims to find the shortest route for a postman who needs to go through every street of a town, and go back to where he started. On a graph, the problem consists in finding a cycle, with minimal length, that passes through every edge at least once. The cycle would be the Eulerian cycle if the graph is Eulerian. If it is not, the problem can be solved by adding the minimum number of edges that make the graph Eulerian, and then finding an Eulerian cycle. An algorithm that constructs the Eulerian cycle in such a case can be an adaptation of the Depth-First Search algorithm. The connection between the concepts of Eulerian graph and depth first traversal is established at different moments, starting with its introduction in the problem of the "Chinese Postman". Then, exercises of application in the theoretical part of the course relate the concepts of Eulerian graph and depth first traversal in the contexts of a weighted graph and a digraph. Additionally, three exercises of the tutorial mobilize the concepts of depth first traversal and Eulerian graph in situations of optimization.

Connections are established with different mathematical topic areas. However, the designers emphasize the most on connections with algorithmic through establishing connections between Eulerian graphs and depth first traversal in many examples and exercises. These exercises represent and model real life situations of optimization, and can be solved using different strategies (heuristic approach, algorithms). Different semiotic representations are mobilized (graphical, figures, language, formal, etc.). For example, "natural language" and graphical representations are used in the introduction to the "Chinese Postman" problem. The cognitive activities of representing and treating could be particularly mobilized for solving the exercises using heuristic approaches. However, conversion to the graphical register is fundamental as it allows mobilizing the Depth-First Search algorithm to find the optimal path in each of the exercises, which can guarantee the reproducibility of the solution in similar situations.

On a more general level, the possibility of using algorithms for solving a large number of the tutorial exercises seems in line with the objective: "recognising algorithmic proofs and applying them in simple problems". In these exercises, identifying the properties of the concepts mobilized determine the choice of algorithms to use and the adaptations to bring to them, if needed. Examples and exercises of "logistic type", that mobilize mathematical processes and algorithms, contribute in making connections between theory and applications, which is aligned with the requirements of the academic path in question. Moreover, the focus on situations of optimization (half the exercises in the tutorial) is aligned with the objective to "know the property on the minimal number of edges of a connected graph". This objective seems closely related to what an engineer in supply-chain management could need in the workplace.

Studies of engineering courses confirm that developing students' ability to make connections is an important aim (Gueudet and al., 2018). Graph theory is a domain of mathematics where many kinds of connections can be established: between and among concepts that relate to mathematical objects and processes; between registers; and with the algorithmic topic area. These connections support a rich mathematical activity that narrows the gap between theory and applications, and develops students' logistic thinking, an essential requirement of their future career in engineering.

References

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on Teaching and Learning Mathematics at the tertiary level: State-of-the-art and looking ahead.* Cham: Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Epp, S. S. (2010). *Discrete mathematics with applications*. Cengage learning.

Flegg, J., Mallet, D., & Lupton, M. (2012). Students' perceptions of the relevance of mathematics in engineering. *International Journal of Mathematical Education in Science and Technology*, *43*(6), 717–732.

Gueudet, G., & Quéré, P.-V. (2018). "Making connections" in the mathematics courses for engineers: the example of online resources for trigonometry. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the Second conference of the International Network for Didactic Research in University Mathematics, INDRUM2018* (pp. 135–144) Kristiansand, Norway: University of Agder and INDRUM.

Gueudet, G., Pepin, B., Restrepo, A., Sabra, H., & Touche, L. (2018). E-textbooks and connectivity: Proposing an analytical framework. *International Journal of Science and Mathematics Education, 16*, 539–558.

Gross, J. L., Yellen, J., & Zhang, P. (Eds.). (2004). *Handbook of graph theory.* Boca Raton, FL: CRC Press.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65–97). New York, NY: Macmillan.

Rosen, K. H. (2012). *Discrete mathematics and its applications* (seventh edition ed.). New York: McGrawHill.

Siemens, G. (2005). Connectivism: A learning theory for the digital age. *International Journal of Instructional Technology & Distance Learning, 2*(1). Retrieved May 2019 from http://itdl.org/Journal/Jan_05/article01.htm

West, D. B. (2001). *Introduction to graph theory* (second edition ed.). Delhi: Pearson education.

DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Allan Tarp

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Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3). Research thus still leaves many issues unsolved after half a century. Inspired by Sartre (2007, p. 20) saying that in existentialism 'existence precedes essence' and by Bauman's (1990, p. 84) sociological imagination, we can ask if mathematics education has a 'goal displacement' seeing its present essence as a goal instead of as an inside means to its outside existing root and goal, mastery of Many?

Mathematics education is based upon textbooks that again are based upon a curriculum for primary and lower secondary school supplemented with side-curricula for upper secondary school. But why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups of students, so why does the number-language?

Both languages have two levels, a language level describing the outside world, and a grammar level describing the inside language. In the word-language, the language level is for all students and includes examples of real-world descriptions, both fact and fiction, whereas grammar level details are reserved for special students. Could it be the same with the number-language, learned by all students through describing fact and fiction? And where grammar level details are reserved to special students?

Also, in contrast to the many letters, words and sentence rules in word-language, a pocket calculator shows that the number-language contains only ten digits and a few operations. And where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. And so are the operations describing counting unbundled, bundles, bundles of bundles where division iconizes pushing away bundles to be stacked in a block, iconized by a multiplication lift, again to be

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 329–334). Paderborn: Universitätsbibliothek Paderborn.

pulled away, iconized by a subtraction rope, to identify unbundled singles that may be placed next-to the stack iconized by an addition cross.

Could it be that the numbering competence that children bring to school contain core mathematics as proportionality, equations and calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction? This would allow designing a curriculum for all students without splitting it up into tracks. And allow the word-language and the number-language to be taught and learned in the same way by describing outside things and actions with inside words and numbers coming from counting and adding.

Before 1970, language was taught as an example of its grammar. Then a reaction came where e.g. Halliday (1973, p. 7) says that “A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a language through its use for communication about outside things and actions.

However, instead of teaching children how to number, the tradition teaches children about numbers and about operations, both to be learned before being applied to the outside world (Bussi & Sun, 2018). Thus, where word-language is taught outside-inside as a description of the outside world, the number-language is taught inside-inside as exemplification of its meta-language or grammar, which makes the number-language more abstract and difficult to learn and to later apply.

So maybe research should go back to the mother Humboldt university in Berlin and reflect on the Karl Marx thesis 11 written on the staircase: “The philosophers have only interpreted the world, in various ways. The point, however, is to change it.”

MEETING MANY, CHILDREN BUNDLE TO COUNT AND SHARE

How to master Many can be observed in a power-free dialogue (Habermas, 1981) with preschool children. Asked “How old next time?”, a 3-year-old will say “Four” and show 4 fingers; but will react strongly if held together 2 by 2: ‘That is not four, that is two twos’, thus insisting that the bundles existing outside should be predicated inside by a ‘bundle-number’ including the unit. Children also use bundle-numbers when talking about Lego blocks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, integrating them next-to each other, they typically say ‘2 7s and 4’.

Children have fun ‘bundle-counting’ their fingers in 3s in various ways: as 1Bundle7 3s, ‘bundle-written’ as T = 1B7 3s, using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2 3s.

Sharing 9 cakes, 4 children take one by turn, and they smile when seeing that '9/4' predicts that they can take a cake twice, thus seeing division by 4 as taking away 4s.

Children thus master numbering and sharing before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the core mastery of Many that children bring to school?

TEXTBOOKS FOR A QUESTION GUIDED COUNTING CURRICULUM

Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied outside. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and two-dimensional blocks with inside bundle-numbers with units. This calls for textbooks from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons; and by comparing with Roman numbers.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H".

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an 'inside bundle-number' describing the 'outside block-number'. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: $T = 2*3 = 3*2$; $T = 2*(3*4) = (2*3)*4$; $T = (2+3)*4 = 2*4 + 3*4$.

Q04, unbundled as decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the learning opportunity is to see that the unbundled occur in three ways: Next-to the block as a block of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q05, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 7/2 means 'from 7, wipe away bundles of 2s'. The calculator says '3.some', thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes a rope dragging away the block to look for unbundled singles. By showing '7-3*2 = 1' the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional learning opportunity is to write and use the 'recount-formula' T = (T/B)*B, saying "From T, T/B times B can be taken away", to predict counting and recounting examples.

Q06, recounting in another unit: "How to change a unit?" Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another learning opportunity is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Q07, recounting from tens to icons: "How to change unit from tens to icons?" Here the learning opportunity is that asking 'T = 2.4 tens = 24 = ? 8s' can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

Q08, recounting from icons to tens: "How to change unit from icons to tens?" Here the learning opportunity is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Q09, finding units: "What are possible units in T = 12?". Here the learning opportunity is that units come from factoring, e.g. 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

Q10, recounting block-sides. "How to recount sides in a block halved by its diagonal?" Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

TEXTBOOKS FOR A QUESTION GUIDED ADDING CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle2 3s = 1*B^2 + 0*B + 2, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Q11, next-to addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

Q12, reversed next-to addition: "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Q13, on-top addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

Q14, reversed on-top addition: "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

Q15, adding tens: "With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Q16, subtracting tens: "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Q17, multiplying tens: "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Q18, dividing tens: "What is 348 recounted in 6s?" Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Q19, adding per-numbers: "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

Q20, subtracting per-numbers: "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Q21, finding common units: "Only add with like units, so how add T = 4ab^2 + 6abc?". Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

DISCUSSION AND FUTURE RESEARCH

So yes, a curriculum for all students is possible without splitting it up into tracks. For the mastery of Many that children bring to school contains core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, as was the communicative turn in foreign language education. However, it can be researched in small scale in preschool, in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological imagination should be welcomed. Quality education, the 4th of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many and welcome children's imaginative alternative to the different versions of mathematics occurring through history.

References

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Bussi, M. G. B. B., & Sun, X. H. (Eds.). (2018). *Building the foundation: Whole numbers in the primary grades, The 23rd ICMI Study.* Cham: Springer.

Habermas, J. (1981). *Theory of Communicative Action.* Boston, Mass.: Beacon Press.

Halliday, M. (1973). *Explorations in the function of language.* London, UK: Edward Arnold.

OECD. (2015). *Improving schools in Sweden: An OECD perspective.* Retrieved from: www.oecd.org/education/school/improving-schools-in-sweden-an-oecd-perspective.htm.

Sartre, J. P. (2007). *Existentialism is a humanism.* London, UK: Yale University Press.

Tarp, A. (2018). Mastering many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education 11*(1), 103–117.

Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford University Press.

TEXTBOOK RESEARCH VIA THE ANTHROPOLOGICAL THEORY OF DIACTICS (ATD): THE CASE OF REPRESENTATION OF FUNCTIONS

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We present a work in progress of a new method for textbook research based on the Anthropological Theory of Didactics (ATD). Representation of the concept of function is chosen as a particular example to validate how the praxeology and levels of didactic co-determination of ATD can be used for textbook research. Two different Norwegian textbooks published at different times are selected. The analyses of the data via relevant theories contributes to the ongoing discussion and addresses recommendations of recent research focusing on mathematics textbooks.

INTRODUCTION

In the 2013 ZDM survey study Fan, Zhu, and Miao (2013) aim to systematically examine, analyse and review relevant research focusing on mathematics textbooks in the previous six decades, and identify future directions in this field of research. They pointed out that there has been a lack of specific research studies centring on the process of development of textbooks, in other words, on how textbooks are produced. In line with the recommendation by Fan et al. (2013), Wijayanti and Winsløw (2017) used a reference model from the Anthropological Theory of Didactics (ATD) as a tool to analyse textbooks. These researchers used the notion of praxeology (fusion of "praxis" and "logos") of the ATD. Praxeology and other elements of ATD are used as a tool to situate the textbook research viewed from a broader perspective, and to develop mathematical textbooks in this work. We illustrate it in this work using the topic of function, which is one of the central topics of mathematics in the middle school. To limit our present work scope for the sake of zooming in to details, we focus on the topic of representation of definition, examples, and exercises of/on functions. The research questions considered here are: How can elements of ATD be used as a tool to analyse how textbooks treat transitions between representations of functions? How do the selected textbooks differ? To answer these questions, the representation of mathematical concepts, specifically, the representation of functions and the transitions between the representations are given due attention in two Norwegian textbooks. Next, the ATD framework and the theories needed are presented.

ANALYTICAL FRAMEWORK

To investigate human mathematical activities, Chevallard (1992) introduced a new mathematics didactic theory, called ATD, that can be used as an epistemological model of mathematical knowledge. It hypothesizes that any activity related to the

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 335–340). Paderborn: Universitätsbibliothek Paderborn.

production, diffusion, or acquisition of knowledge should be interpreted as an ordinary human activity. Hence, ATD enables researchers to study human mathematical activities. Chevallard modelled the activities via the practice and knowledge block. The practice block, as the minimum human activity, consists of a task (T), and technique (τ) used to achieve the task. Hence, the practice block is a tuple (T, τ). The knowledge block is made of a technology (θ) to justify the technique (τ) and further the technology itself is explained with a wider discourse called theory Θ. The knowledge block, in turn, is a tuple (θ, Θ), together the theory of praxeology is a quadruple (T, τ, θ, Θ) and this can model an amalgam of human practice and knowledge (Winsløw, 2011; Putra, 2017; Wijayanti & Winsløw, 2017). Readers are referred to Tesfamicael and Lundeby (2019) for an expanded presentation.

Let us give a particular example of such modelling within the topic of the teaching of relations and functions. Consider that students are given the task (T) of representing a quadratic function. The different ways to represent such a function, symbolic, using table or graph can be used, and these are called techniques (τ). The explanations related to the techniques can be considered as a technology (θ), while the whole discourse about representation can be viewed as Theory (Θ).

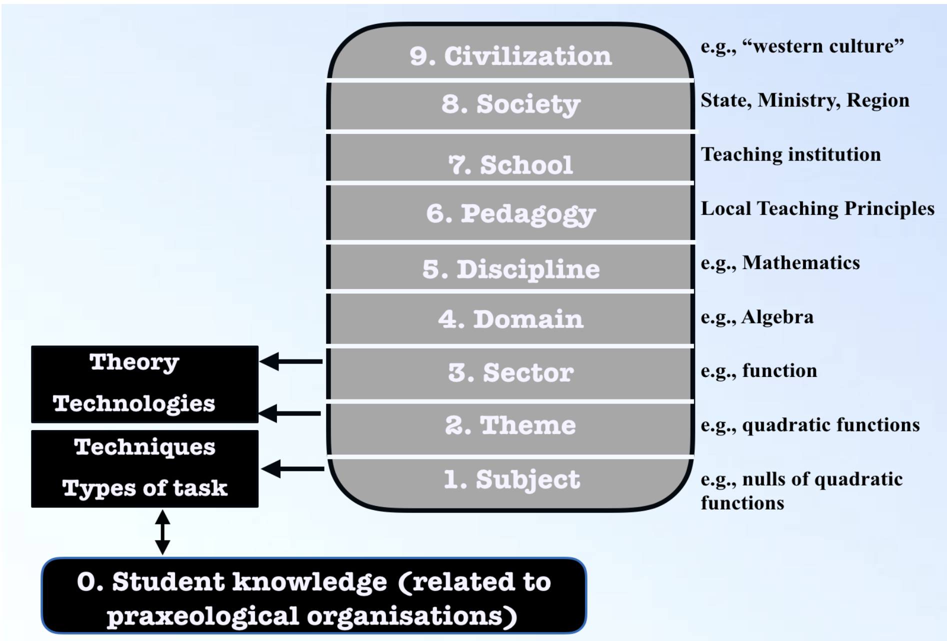

Figure 1. Levels of didactic co-determination as proposed by Chevallard (2002) and adopted for this work.

The praxeology model cannot be fully grasped without situating it within the broader context of a whole hierarchy of institutional levels (Figure 1) as presented by Chevallard (Winsløw, 2011). Notably, the development and use of the textbook are influenced by different factors. The culture and society (level 9 and 8, respectively) set the policy and general direction, while publishers develop textbooks according to the preferences of schools (at level 7) who decides which textbook can be used in a classroom. So, it is not only via the pedagogy content (level 6) and the subject matter content (level 5) that decisions about what goes into the textbook development are made but other levels too.

Web of Representations

Representations are vital in understanding and communicating concepts. (Duval, 2006, p. 103) defines representation as "something that stands for something else" and argues further that an object can have multiple representations and argued that what matters are not the representations but their transformations, because mathematical processing always involves substituting some semiotic representation for another" (p. 107). If the activity or the task (T1) considered here is representing the concept function, the different semiotic representations are termed as techniques (τ) in ATD as described above. Different representations peculiar for functions and the translation between them was given by Janvier (1987). An expanded form consisting of seven different representations called web of representations are provided in Elementary and Middle school mathematics (Van et al., 2015), see Figure 2.

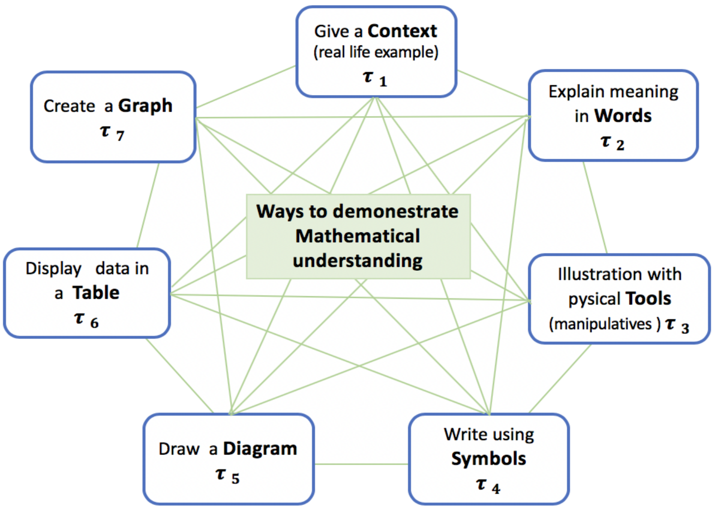

Figure 2. Web of representations together with 7-technique (τ) (adopted from Van et al. 2015, p. 45).

To clarify some of these representations further, *words* includes the plain usage of the (English) words and numbers, while *physical tool* means the involvement of manipulatives, concrete or virtual, things in the example or exercises, not just drawings and mentions of them.

Transition between the representations

Understanding the concept of function from cognitive perspectives implies an ability to make connections between different representations of the concept (Siti, 2010). Lesh et al. (2003) indicated that students who have difficulty in translating a concept from one representation to another have difficulty in applying the concept in problem solving and computations. Textbooks should provide an opportunity for learners to utilize different representations of mathematical concepts.

METHOD

Document analysis is the method used in this work. In order to choose which textbooks to analyse, a simple survey was performed. We asked a question to teachers about the mathematics textbooks they use for the topic relations and functions in Norway. Our question was: *which textbook do you use to teach the topic function*? It was posted/shared in a Facebook group of around 10,000 teachers. 119 teachers responded, and the result is Faktor 25.2%, Grunntall 22.7%, Maximum 13.4%, Tetra 12.6%, Nummer 7.6%, Mega 5.9%, Sirkel 4.2%, and others 8.4%. In this study, only two textbooks are considered. These are Faktor, since many use it, and Maximum for it is one of the most recently published textbooks. Specifically, we analyse how the topic of function is treated in Faktor 2 (Fakt2), Maximum 9th grade (Max9) and Maximum 10th grade (Max10).

In the selected textbooks, the techniques applied in the examples and exercises, which deal with the function concept, are counted, analysed, categorized, and compared further. In addition, the transitions from each representation to another is modelled as an activity under the ATD framework. So, another Task (T2) is defined as the use of transitions between the representations. The techniques applied are defined as follows: $\tau\_ij \rightarrow$ *as transition from representation technique* τ_i to τ_j. The possible transition between the 7-techniques, τ_i, *where* $1 \leq i \leq 7$, of representation used above in Figure 3 makes up a 7 x 7 matrice, without the τ_ii. The Textbooks considered are also scrutinized using this transition, which transitions among the 42 techniques of the transitions τ_ij, where $1 \leq i \leq 7$ exist?

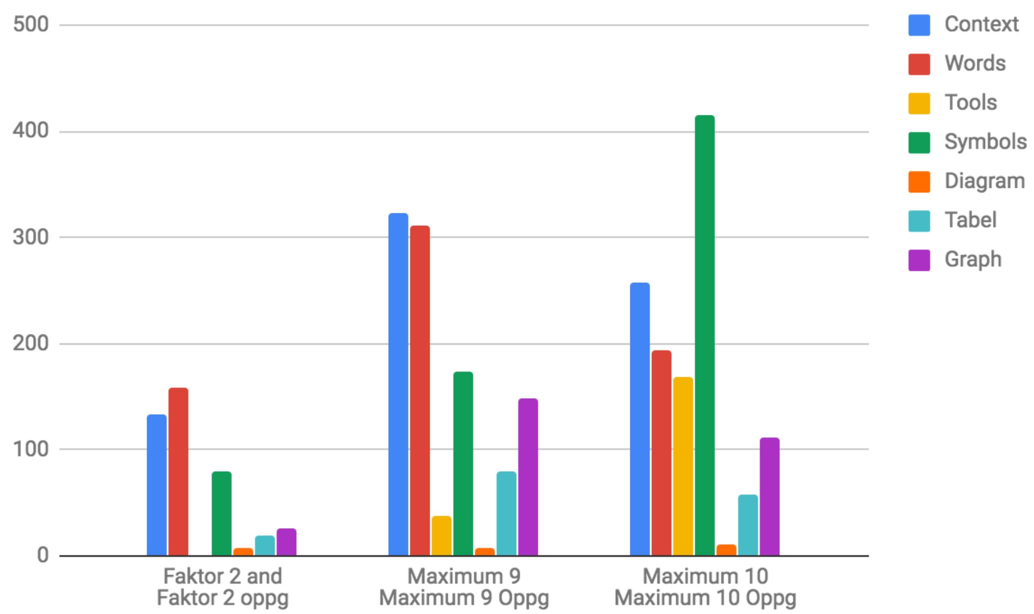

Figure 3. Diagram showing the statistics of the different representations (i.e., number of *words, physical tools*, *symbols*, *diagrams*, *tables*, and *graphs*).

DATA ANALYSIS

The textbooks selected were analysed using the web of representation, seven different *techniques* (τ) described above*,* for the concept function via the praxeology model described above. The number of the different *techniques* (τ_i), where i ≤ 7, are counted and analysed. The summary is given in Figure 3.

The number of transitions, from a particular representation to another one are counted and analysed. If real life context is involved, for example described by *words* then we counted it as *context* (τ_1) and but if there is an algebraic expression mixed with *context,* we count the transition from *symbols* (τ_4) to the other representation. *However, if a task is not about transition between representations, we do not count it in the analysis.* The results are summarized as follows for the transitions *from representation technique* τ_i *to* τ_j*,* of transitions technique τ_ij.

Transitions	τ_12	τ_14	τ_16	τ_17	τ_24	τ_27	τ_37	τ_41	τ_42	τ_45	τ_46	τ_47
Faktor	27	29	4	9	0	19	0	0	37	0	22	21
Maximum	39	23	12	22	41	10	7	1	131	6	1	41

Transitions	τ_56	τ_61	τ_62	τ_64	τ_67	τ_71	τ_72	τ_74	τ_75	τ_76
Faktor	0	0	5	1	3	1	25	2	0	0
Maximum	1	4	26	15	20	30	122	19	1	1

Table 1. Among the 42 possible transitions only 22 of them happened in Faktor 2 and Maximum grade 9 book.

DISCUSSION AND CONCLUSION

In both the textbooks, Fakt2 and Max9, *word* (τ_2) *and symbol* (τ_4) representations dominate, while *symbols* (τ_4) representation is visibly higher in Max10 which makes sense since the abstraction level in grade 10 should be higher than in grade 9, in general. But the textbooks differ when it comes to the representations in contexts, tools, and graphs. The use of context is more pronounced in Maximum than Faktor. It also seems that the use of manipulatives (physical or virtual) is very small in Faktor compared to Maximum. This could be that the latter is published in recent times. Using this analysis can help the development of textbooks by taking into consideration which relevant representations for the topic are under-represented or missing.

Among the 42 possible transitions from one representation to another, only 22 are registered. There is no even distribution of the different transitions. A few of the transitions dominate in the textbooks analysed. These are τ_42, τ_72, τ_12, τ_47 and τ_14. This means that transitions from and to symbols are the most common for the function concept in both textbooks. In the transition from word to a symbol, from symbol to table, and graphs to words, the textbooks differ remarkably. It could have been interesting to create examples and tasks that can foster the use of different representations and more transitions to deepen the understanding of the mathematical concept in consideration. The praxeology of ATD can help to model the activities in a textbook. At this stage, we are able to identify which representations and transitions between them are used most and which are not via the practice block of praxeology. We are looking for ways how the limitations can be improved and even by approaching the textbook development from the perspective of institutional levels model of ATD**.**

References

Chevallard, Y. (1992). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. In R. Douady & A. Mercier (Eds.), *Research in Didactique of Mathematics, Selected Papers* (pp. 131–167). Grenoble: La Pensée Sauvage.

Chevallard, Y. (2002). Organiser l'étude 3. Écologie & régulation. In J. L. Dorier et al. (Eds.), *Actes de la 11e école de didactique des mathématiques*. Grenoble: La Pensée Sauvage.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education, 45*(5), 1–14.

Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesh, R., Cramer, K. A., Doerr, H., Post, T., & Zawojewski, J. (2003). Model development sequences: Models and modelling perspectives on mathematics. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics* (pp. 35–58). Mahwah, NJ: Lawrence Erlbaum Associates.

Putra, Z., & Witri, G. (2017). Anthropological theory of the didactic: A new research perspective on didactic mathematics in Indonesia. In S. Syahrial, S. Saharudin, & J. Marzal (Eds.), *Proceedings of the second international conference on education, technology, and sciences: Integrating technology and sciences into early and primary education* (pp. 142-149). Universitas Jambi, Indonesia: Fakultas Keguruan dan Ilmu Pendidikan.

Siti A. A. (2010). Comprehending the concept of functions. *Procedia – Social and Behavioral Sciences, 8*, 281–287. http://doi.org/10.1016/j.sbspro.2010.12.038.

Tesfamicael, S. A. & Lundeby, Ø. A. (2019). A Comparative study of norwegian and ethiopian textbooks: The case of relations and functions using Anthropological Theory of Didactics (ATD). *Universal Journal of Educational Research 7*(3), 754–765. https://doi.org/10.13189/ujer.2019.070315.

Van, W. J. A., Karp, K. S., & Bay-Williams J. M. (2015). *Elementary and middle school mathematics: Teaching developmentally*. Harlow, UK: Pearson Education limited.

Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *REDIMAT, 6*(3), 307–330. https://doi.org/10.1783/redimat.2017.2078

Winsløw, C. (2011). Anthropological theory of didactic phenomena: Some examples and principles of its use in the study of mathematics education. In M. Bosch Casabo, J. Gascón, A. Ruiz Olarría, M. Artaud, A. Bronner, Y. Chevallard, G. Cirade, C. Ladage, & M. Larguier (Eds.), *Un panorama de la TAD* (pp. 117–140). Barcelona: Centre de Recerca Matemàtica.

50 YEARS OF REALISTIC MATHEMATICS EDUCATION IDEAS AND THEIR IMPLEMENTATION IN TEXTBOOKS — THE CASE OF ADDITION AND SUBTRACTION

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The Dutch reform movement towards Realistic Mathematics Education (RME) started at the end of the 1960s. As an alternative for the then prevailing mechanistic approach, tasks, lessons and longitudinal teaching sequences were developed together with suggestions for helpful contexts and didactical models. Over the years, RME ideas were laid down in many studies, covering both general ideas on the teaching of mathematics, and elaborations in specific subdomains. Simultaneously, these ideas were implemented in consecutive generations of mathematic textbooks. A systematic analysis of RME documents and textbooks with the focus on addition and subtraction revealed that Dutch textbooks to a certain degree have evolved along the lines of RME, but in some way they also deviate from RME ideas.

INTRODUCTION

The starting point of the development of Realistic Mathematics Education (RME) (Van den Heuvel-Panhuizen & Drijvers, 2014) was the establishment of the Wiskobas project in 1968. Wiskobas is an acronym for *Wiskunde op de Basisschool*, which means mathematics in primary school. The main goal of Wiskobas was to elaborate an alternative for the—at that time in the Netherlands—prevailing mechanistic mathematics education. This approach focused on demonstrating fixed procedures in a step-by-step manner, starting directly at a formal level and with little or no attention for developing insight. In contrast, Wiskobas emphasized that students should get the opportunity to realize what happens in a mathematical situation and should be supported to imagine what could happen (De Jong, Treffers, & Wijdeveld, 1975; Treffers & Van den Heuvel-Panhuizen, in press). To this end, students were presented meaningful context situations based on which they could develop new mathematical concepts. Although this and other Wiskobas ideas are still characteristic of RME, it is not a fixed theory; over the decades, different emphases have been made (Van den Heuvel-Panhuizen, 2019).

In the current study, we investigated how initial RME designs and their underlying ideas evolved since the onset of RME and how these designs in the course of the years appeared in textbooks. To clearly bring into view the actual meaning of RME when it is laid down in instructional material, we focused on a particular subdomain: addition

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 341–346). Paderborn: Universitätsbibliothek Paderborn.

and subtraction in the lower grades (before written algorithmic calculations are introduced). Our research questions were: *How did the RME approach on addition and subtraction in Grades 1, 2, and 3 evolve since the early days of RME? How was this approach implemented in consecutive generations of RME-based textbooks?* In this paper, we report on some preliminary results of this study.

METHOD

To answer our research questions, we started carrying out a systematic analysis of RME curriculum documents and of RME-based textbooks.

Selection of RME curriculum documents

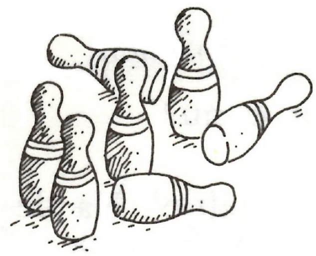
Because of the overwhelming amount of literature on RME, we made a selection of core curriculum documents that either provide an overarching overview of RME ideas or specifically address addition and subtraction. These documents include the first *Wiskobas* overview of primary school mathematics education (De Jong, Treffers, & Wijdeveld, 1975), the articulation of RME teaching principles (Treffers, 1987; Van den Heuvel-Panhuizen, 2001), and overviews of goals and teaching-learning trajectories including addition and subtraction as laid down in the so-called *Proeve* (Treffers & De Moor, 1990) and *TAL* (Van den Heuvel-Panhuizen, 2008) documents.

Selection of textbooks

Since the 1980s, textbooks have been published that were influenced by the ideas of Wiskobas, later known as RME. The selection of textbook series to be included in our study was based on the following two criteria. First, the design of a textbook should be intentionally RME-based, which, for example, is clear if in the teacher guidelines explicitly references are made to Wiskobas or RME. Textbooks that are adapted editions of textbooks from other countries may have some RME characteristics but were not considered as RME-based. The second criterion concerns the market share. Since a larger market share is an indication of a greater acceptance of the RME-based textbook by teachers, we only included textbooks that reached at some point fifteen percent market share or more. The two criteria combined resulted in a collection of eleven (editions of) textbook series in use from the 1980s on until today: four consecutive editions of *De Wereld in Getallen* [The World in Numbers], three editions of *Pluspunt* [Plus Point], the textbook series *Rekenen & Wiskunde* [Arithmetic & Mathematics] and its successor *Wis en Reken* [Certainty and Calculate], and two editions of *Rekenrijk* [Rich Arithmetic / Kingdom of Arithmetic].

Analysis procedure

The first step of our study consisted of the analysis of the RME core curriculum documents. We detected all suggestions offered in these documents for facilitating the learning of addition and subtraction, the underlying ideas regarding the intended teaching-learning process, and the related RME teaching principles. For example, a learning facilitator mentioned in *Wiskobas* (1975) is the use of arrow language (Fig. 1). This arrow language is used before the equal sign is introduced, as a symbol of "something is happening" in an addition or subtraction situation (ibid., p. 39). The underlying ideas described are threefold: (1) students explain in their own words what is happening and write this down themselves using informal arrow language; (2) arrow language is used in all kinds of situations, providing multiple meanings of addition and subtraction; and (3) from the start on, the relationship between addition and subtraction comes to the fore. In these ideas, several of the later formulated RME teaching principles (Treffers, 1987; Van den Heuvel-Panhuizen, 2001) can be recognized, especially the activity principle – using students' input in the teaching-learning process, and the reality principle – meaningful situations are used as a starting point to help students realize and understand the mathematical constructs involved.



Els: "Of the seven pins, I overthrew three."

$7 \xrightarrow{-3} 4$

Wim: "I already put up four pins and I still had to put up three more."

$4 \xrightarrow{+3} 7$

Figure 1. Arrow language in Grade 1 (De Jong et al., 1975, p. 40)

The results of this first analysis were, next to answering the first research question, used to develop a framework for the analyzation of the textbooks included in our study. In this framework, we incorporated all inventoried learning facilitators and their intended use. Table 1 shows a part of this analysis framework.

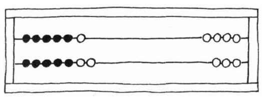
RME learning facilitator	Intended use of this learning facilitator
Bus context 	• Playing getting on and off the bus gives meaning to addition and subtraction • Simultaneous introduction of addition and subtraction in one situation
Arrow language	• Used with various situations, providing multiple meanings of addition and subtraction • Students explain situations in their own words and write this down with arrow language • Relationship is laid between addition and subtraction
Arithmetic rack 	• Providing number images based upon the tens, fives, and double structures • Replacing counting by structured calculation
Number line	• Locating numbers in the number row • Supporting structured calculation

Table 1. Part of the analysis framework for textbooks. Sources of the pictures: *Wiskobas* (1975, p. 41); *Proeve* (1990, p. 45)

Next, for each textbook series we selected those parts in which, according to the teacher guidelines, instruction and whole-class teaching is supposed to take place. Then we searched in these student materials and teacher guidelines for the presence of learning facilitators, and for indications of their intended use. This search was done twice to ensure that no relevant information was overlooked. Additionally, similar information was searched for in other textbook materials such as user brochures.

SOME PRELIMINARY RESULTS

Evolvements in the RME approach to teaching early addition and subtraction

What (among other things) is present in all core RME documents is the use of meaningful situations as a starting point, the use of models, and the use of students' own productions as input in the teaching-learning process. More in particular, learning facilitators such as the bus context, arrow language and the number line are present in most core documents, although there are changes over time.

A first change that occurs concerns the use of the bus context and the arrow language. In the *Wiskobas* (1975) document, emphasis was put on the simultaneous introduction of addition and subtraction in one situation, the relationship between the two operations, and that arrow language is used in all kinds of situations. These three points are no longer mentioned in the descriptions in the more recent documents.

Another change is the introduction of the arithmetic rack for making number patterns (i.e., visualizing sets and subsets of numbers) and calculating up to 20. This learning facilitator was not yet present in *Wiskobas* (1975), but was introduced later in the *Proeve* (1990). In that publication and thereafter, it is described that the arithmetic rack is used to provide number patterns based upon the tens, fives and double structures, and to use these to replace calculation by counting with structured calculation.

Also, the number line has undergone changes. Here the changes were in its appearance. In the *Wiskobas* (1975) document, segmented number lines were used, while in the later published *Proeve* (1990) and *TAL* (2008) documents empty number lines were presented. This change in appearance has significant implications for the intended use of the number line. For example, to position numbers on a segmented number line (Fig. 2, left) students can simply count the tick marks, whereas a line with no or little markings (Fig. 2, right) evokes the use of number relations (e.g., 47 is nearby 50 and 25 is in the middle between 0 and 50). In other words, while the older segmented number line may lead students to continue using counting procedures, the newer empty number line stimulates students to use shortened and structured procedures.

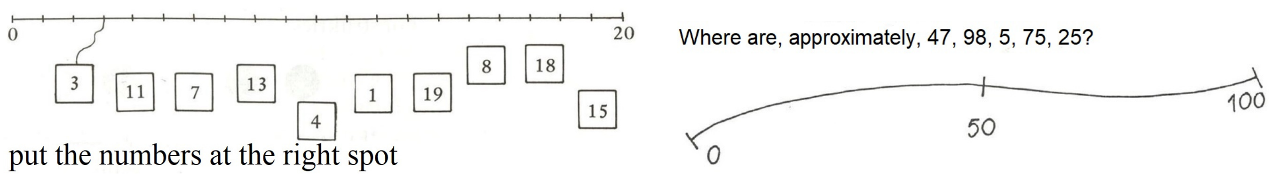


Figure 2. Positioning numbers on a segmented number line in *Wiskobas* (1975, p. 44) (left) and an almost empty number line in *Proeve* (1990, p. 52) (right)

Implementation of the RME approach on addition and subtraction in textbooks

The learning facilitators discussed in the previous section, are in principle all present in the analyzed textbook series, but in several ways they also deviate from how they are presented in the core documents. For example, all eleven textbooks use the bus context to introduce addition and subtraction, but in none of the textbooks addition and subtraction take place simultaneously in one and the same situation. In some of the teacher guidelines this is stated explicitly by saying that at each bus stop only one of the operations happens: people are either getting on or off the bus.

Also, regarding the use of arrow language the interpretation in the textbooks differ from that in the core documents. Instead of letting students actively write down arrow language themselves and using this students' input, in all textbooks arrow language is offered to the students who only have to fill in the missing numbers (Fig. 3) or operation sign. Even if students have to write down a problem situation in arrow language themselves, the guidelines state that the teacher has to demonstrate each step. As a consequence, the idea of giving students access to the formal world of mathematical symbols by letting them create their own context-based informal language to describe situations mathematically is completely lost.

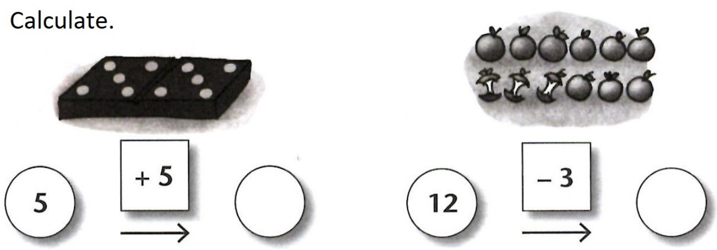


Figure 3. Arrow language in *Pluspunt*, 3rd edition, Grade 1, student book 7-8, p. 46

Furthermore our analysis revealed that applying the arrow language in all kind of problem situations, which was emphasized in the *Wiskobas* (1975) document, but did not show up in the *Proeve* (1990) and *TAL* (2008) documents, is still used in all current textbooks. A similar finding concerns the kind of number line that is used. As mentioned before, in *Wiskobas* (1975) the number line was segmented and in *Proeve* (1990) and *TAL* (2008) it became an empty number line. Nevertheless, nowadays in the textbooks for positioning numbers up to 20 still segmented number lines are used.

Finally, the arithmetic rack that was introduced in the *Proeve* (1990), is implemented in all (nine) textbooks that have been published since. This is done consistent with the use described in the *Proeve* (1990) and *TAL* (2008) (see Fig. 4 for an example).

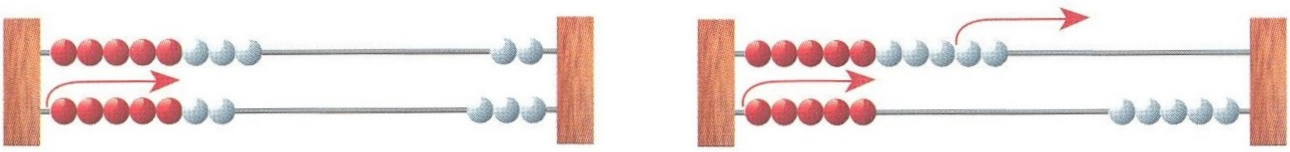


Figure 4. Two ways to calculate 15 – 7 on the arithmetic rack using number patterns in *De Wereld in Getallen*, 4th edition, Grade 2, student book 4A, p. 29

CONCLUSIONS

Our study illustrates that RME ideas have developed over time. Learning facilitators were not always conceptualized in the same way. An example of this is the number line.

Regarding the implementation of RME ideas in textbooks, so far, our study shows a diverse picture. Although all RME learning facilitators are incorporated in the textbooks we analyzed, this is not always done in accordance with RME ideas. A clear example of this is the way arrow language is used in textbooks, giving way less room for students' input than originally was intended.

References

De Jong, R., Treffers, A., & Wijdeveld, E. (Eds.). (1975). Overzicht van wiskundeonderwijs op de basisschool [Overview of mathematics education in primary school]. *Wiskobas Bulletin*, 5(2/3).

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics instruction – The Wiskobas project*. Dordrecht, the Netherlands: Reidel.

Treffers, A., & De Moor. (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool 2*. [Design of a national program for mathematics education in primary school 2]. Tilburg, the Netherlands: Zwijsen.

Treffers, A., & Van den Heuvel-Panhuizen, M. (in press). Dutch didactical approaches in primary school mathematics as reflected in two centuries of textbook. In M. van den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics – Teaching and learning in the context of Realistic Mathematics Education.* Cham, Switzerland: Springer.

Van den Heuvel-Panhuizen, M. (2001). Realistic mathematics education in the Netherlands. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching. Innovative approaches for the primary classroom* (pp. 49–63). Buckingham, Philadelphia: Open University Press.

Van den Heuvel-Panhuizen, M. (Ed.). (2008). *Children learn mathematics. A teaching-learning trajectory with intermediate attainment targets for calculation with whole numbers in primary school.* Rotterdam, the Netherlands: Sense.

Van den Heuvel-Panhuizen, M. (2019). Didactics of mathematics in the Netherlands. In W. Blum et al. (Eds.), *European Traditions in Didactics of Mathematics* (pp. 57–94). Cham, Switzerland: Springer.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 521−525). Dordrecht, Heidelberg, New York, London: Springer.

A LESSON STUDY INTERVENTION TO DEVELOP PRIMARY SCHOOL STUDENTS' ABILITY TO PERFORM MENTAL FOLDING OPERATIONS'

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I designed a lesson study to develop students' ability to perform mental operations and to clarify the effects through actual classroom-based lessons. From observing the lesson, I found that the strategy of the mental operation used to assemble cubes could be applied to imaginarily construct the more difficult cuboids from the flat patterns of rectangles. In addition, the children who had experience using this mental strategy for other problems and recognizing its usefulness began to independently apply it to similar problems, showing that they had acquired spatial ability skills.

BACKGROUND

Spatial reasoning plays an important role in predicting overall mathematics success with even greater predictive power than general mathematics score (Farmer et al., 2013). The ability to mentally rotate objects in space has been singled out by cognitive scientists as a central metric of spatial reasoning (see Jansen, Schmelter, Quaiser-Pohl, Neuburger, & Heil, 2013; Shepard & Metzler, 1971). However, this is a particularly undeveloped area of the current mathematics curricula in Japan. Primary school children in grade four in Japan learn about shifting dimensions between the 2-D flat patterns of squares and rectangles and the 3-D solid patterns of cubes and rectangular cuboids (henceforth, "cuboids") in their arithmetic textbook. However, the ability to perform mental operation has not been well studied in lesson study interventions. I assume that if students find a mental strategy to choose the bottom and move or rotate other faces and notice its usefulness when doing other spatial tasks, they will develop mental operation skills by themselves. Accordingly, the question arises of how students' mental operations can engage in, and benefit from classroom-based interventions of mental folding tasks. Thus, the research questions are as follows: Is it appropriate for students to choose the bottom face as a strategy to imagine mental folding tasks? And Is there an increase in students' practical use of this spatial strategy?

THEORETICAL FOUNDATIONS

Spatial ability has been conceptualized in a multidimensional fashion consisting of several separate but correlated skills. Two broad categories of multidimensional models have emerged: one based in the psychometric tradition (Carroll, 1993) and the other in more theoretically driven models (e.g., Uttal et al., 2013). This study adopted a theoretical model proposed by Uttal and colleagues (Newcombe & Shipley, 2015;

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 347–352). Paderborn: Universitätsbibliothek Paderborn.

Uttal et al., 2013). Mental operations are one kind of spatial reasoning that is classified as an intrinsic-dynamic skill (Uttal et al., 2013). Dynamic spatial skills involve transforming an object or set of objects, perhaps by rotating, folding, bending, or scaling (Davis & Spatial Reasoning Study Group, 2015). A meta-analysis by Uttal et al. (2013) found that training interventions, regardless of age, resulted in a significant improvement to intrinsic-dynamic spatial reasoning (effect size 0.44). Catherine et al. (2015) studied learning environments that foster the development of students' (aged 4–8 years) mental rotation skills and found associated gains in their overall mathematics performance over four months. However, there are few studies of age-appropriate elementary school mathematics lesson study interventions designed to develop student's ability to perform such mental operations, and on detailed lesson records concerning the strategies that children have acquired to improve their spatial reasoning ability.

RESEARCH METHOD

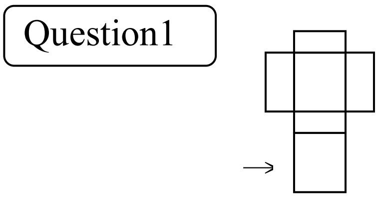
The lessons' design

The findings of previous research noted that it is easier for children (aged 7–12) to imagine and fold the unfolded plane of a cube than a cuboid (Watanabe, 2010). For this reason, I designed two lessons to develop their spatial abilities using mental folding. This is in the reverse order from the textbooks.

1. The first lesson was to determine to where to place the missing face from a flat cube patterns so as to assemble it correctly. The students choose a bottom face and imagine folding four faces, and thus look for the position of an upper bottom. Question 1 will be easier for students than 2, because if students choose a bottom face at the center, they will imagine folding two sides together. However, the faces of question 2 require imagining three or four side faces around the bottom face.

Figure 1. Paper-folding tasks. Students imagine and assemble the flat pattern and look for the upper face.

2. The second lesson was to figure out which two sides of the flat pattern of a cuboid will border each other when assembled. The children will imagine folding faces and look for overlapping sides. Question 1 will be easier for students than 2, because a face that includes an arrowed side stands straight on the bottom. However, the face of question 2 will bend and thus change the direction.



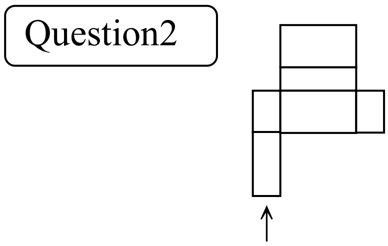


Figure 2. Paper-folding tasks. Students imagine and assemble this flat pattern, and look for a border on the arrowed side.

Data collection

A total of 32 fourth graders (aged 10) participated in these two lessons. I recorded the state of talk and discussion of children in the class for the following purpose. What kind of strategies did students present and how did they discuss the usefulness of these strategies. I also collected a copy of each students' notes for two reasons: First, the notes were used to verify the strategy they used for question 1. Second, the students' notes were used to clarify the practical use of the strategy for question 2. I conducted the two lessons and observed how students thought and worked in both of them.

Pre- and post- tests

The pre-test (12 items) and post-test (16 items) were conducted through a paper-pencil test. The questions included four types of paper-folding tasks, "looking for a missing face" and "which sides will border each other" using cube and cuboid flat patterns. I investigated how much the strategy acquired in the lesson helped the students in the post-test survey.

RESULTS

The first lesson was to determine where to place the missing face from a flat cube pattern so as to assemble it correctly. The students solved question 1 and described how to solve it in their notebooks. After solving it by themselves, some of the students explained their strategies to their classmates. Three strategies were shown as follows.

Strategy 1 (used by 13 students)

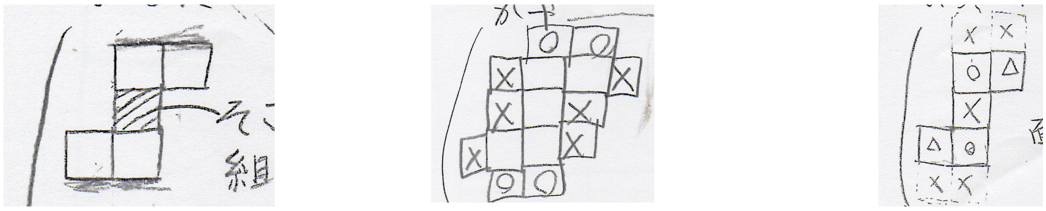
Choose the bottom face and assembled its four side faces around. Then I can find four sides that would be attached to the last upper bottom face.

Strategy 2 (used by two students)

Put one upper bottom face to every side of the unfolded planes one by one. If the bottom face overlapped or did not bend, these positions were mistakes.

Strategy 3 (used by one students)

The upper and bottom face in parallel positions will line up one face and skip when it is unfolded. Then choose a bottom face and look for the skipped position.



strategy1 **strategy 2** **strategy 3**

Figure 3. Strategies used by students to find the missing face.

After they presented their strategies, students discussed which strategy was useful.

Kazuki said to Kaya (who presented strategy 2) “It takes too much time.” Kaya answered. “I think so now.”

Makoto said to Hiroki (who presented strategy 3) “It’s easy to find two parallel faces, but the remaining two faces are difficult to imagine.” Hiroki said. “It’s easy for me because I learned the location of the parallel faces.”

Ane said to Hinata (who presented strategy 1) “I do it the same way. It’s easy to imagine.”

After this discussion, the teacher gave question 2 to the students and said. “You choose the strategy that is easiest to imagine.” Out of 33 students, 23 chose strategy 1 and obtained the correct answer.

The second lesson was to figure out which two sides of the cuboid’s frat pattern will border each other when assembled. In this lesson, the students searched for bordering edges by looking for sides of the same length. After solving this independently, some of the students explained their strategies to their classmates. Two strategies were demonstrated.

Strategy 1 (used by 23 students)

Choose the bottom face and assembled the other faces around. Then I can find the side that would border the arrowed side.

Strategy 2 (used by two students)

I connected the sides of the same length with a pencil line.

I observed how the students solved the problem. Many students applied a strategy that involved choosing a bottom face to start and manipulated the remaining four sides to ensure that the sides would overlap with each other when it was assembled as a cuboid. After applying this method, the children connected sides of the same length and checked whether the answer was correct.

Post-test

After the two lessons, I conducted a post hoc survey to investigate in which lesson students had acquired the strategy. In total, six of the eight students, who had a low score in the pre-test assessment improved in the post-test assessment. Considering the average score overall, they earned good scores on not only the same types of paper-folding problems but also on types of paper-folding problems they had not learned. Based on these results, it was possible to infer that they had acquired experience in applying a learned spatial reasoning strategy to other problems, and understood the usefulness of the mental folding operations and the spatial reasoning strategies that associated with it. They acquired experience applying a learned strategy to other problems and realized its usefulness in different contexts. Through learning the strategy, they also became able to apply it to problems in the post-test that had not been previously studied.

CONCLUSION

The results of this study found that the strategy of mental operations used to assemble cubes could be applied to mentally construct the more difficult cuboids from a flat pattern. In addition, the students who had experience of using this mental strategy for other problems recognized its usefulness and began to apply it independently to similar problems, thus showing that they had acquired spatial ability skills.

References

Carroll, J. B. (1993). *Human cognitive abilities: A survey of factor-analytic studies.* Cambridge, UK: University of Cambridge Press. https://doi.org/10.1017/CB0978051157312

Catherine, D. B., & Zachary, H. (2015). The role of 2D and 3D mental rotation in mathematics for young children: What is it? Why does it matter? And what can we do about it? *ZDM Mathematics Education*, 47(3), 331–343.

Clements, D. H., & Sarama, J. (2003). *Engaging young children in mathematics.* New York: Routledge, (267 -297).

Davis, B. & Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years*. New York: Routledge.

Farmer, G., Verdine, B., Lucca, K., Davies, T., Dempsey, R., Newcombe, N., et al. (2013). Putting the pieces together: Spatial skills at age 3 predict spatial and math-performance at age 5. Seattle: SRCD poster presentation.

Jansen, P., Schmelter, A., Quaiser-Pohl, C., Neuburger, S., & Heil, M. (2013). Mental rotation performance in primary school age children: Are there gender differences in chronometric tests? *Cognitive Development*, *28*, 51–62.

Shepard, R. N., & Metzler, D. (1971). Mental rotation of three-dimensional objects. *Science*, *171,* 701–703.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). *The malleability of spatial skills: Mind, cognitive science, and human experience.* Cambridge, MA: MIT Press.

Wai, J., Lubinksi, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817.

Watanabe, S. (2010). Research on how the elementary school pupil imagines a solid body from its unfolded plane. *Journal of Japan Society of Mathematical Education, 92*(10), 12–23.

RATIO, RATE AND PROPORTIONAL RELATIONSHIPS IN JAPANESE CURRICULUM MATERIALS

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Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn. In this study, Japanese curriculum materials' treatments of these topics were investigated. Both vertical and horizontal analyses (Charalambous et al., 2010) were conducted, examining when and what specific topics are discussed and how they are treated in the Japanese curriculum materials.

INTRODUCTION

Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, in the United States, and perhaps in other countries, both teachers and students find these topics challenging. Japanese students have consistently performed near the top of the world in many international studies. One important factor in students' opportunities to learn is the textbooks that are used. Kilpatrick, Swafford, and Findell (2001) pointed out, "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36). Therefore, understanding how Japanese textbooks treat these topics may be of interest to mathematics educators outside of Japan. The goal of this study is to examine how one of the most widely used textbook series in Japan introduces and develops these topics.

RATIO, RATE AND PROPORTIONAL RELATIONSHIP

In Japanese, the term *hi* corresponds to the English term ratio, and *hirei* and *hirei no kankei* corresponds to proportion and proportional relationship, respectively. However, there is no Japanese word that directly corresponds to the term rate. The Japanese word, *wariai*, can be translated into ratio, rate or even proportion, depending on the context. According to Tabata (2010), *wariai* was an everyday Japanese term that was adopted into school mathematics after World War II, in part so that it could refer to related yet distinct ideas about ratio. However, the ambiguity of the term *wariai* may actually be due to the ambiguity of the terms ratio and rate in English. There appear to be different interpretations of ratio and rate. Some consider a ratio as a multiplicative comparison of two quantities from the same measure field while a rate is a comparison of two quantities from distinct measure fields. Thus, ratio and rate are mutually exclusive. However, others consider a rate as a special ratio where two quantities being compared come from different measure spaces while a ratio is a comparison of any two quantities. Still others consider a ratio to be a comparison of two non-varying quantities while a

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 353–358). Paderborn: Universitätsbibliothek Paderborn.

rate represents a proportional relationship between co-varying quantities. For the purpose of this research, we considered the union of these different interpretations of ratio and rate as the Japanese curriculum materials were analysed.

RESEARCH QUESTIONS

This content analysis study examined the following two research questions.

RQ 1: What specific topics related to ratio, rate and proportional relationships are discussed in the Japanese curriculum and in which grade, or grades?

RQ 2: How are the ideas related to ratio, rate and proportional relationships introduced and developed, both within and across grades, in the Japanese curriculum materials?

METHODOLOGY

Curriculum materials

There were two different data sources for the current study. One data source consists of documents published by the Japanese Ministry of Education, Culture, Sports, Science and Technology (MOE). The MOE publishes the National Course of Study (COS), which specifies what topics are to be taught in what grade level. The MOE also publishes a second document, often called *Teaching Guide*, that explains and elaborates the specific standards in the COS. The other data source was the textbook series published by Tokyo Shoseki, one of the most widely used series in the elementary and lower secondary schools. This particular series has been translated into English (Fujii & Iitaka, 2012). Thus, even though the analysis was conducted on the original Japanese version, the English version will be used whenever aspects of this textbook series are discussed.

Data analysis

This study employs both vertical and horizontal analysis (Charalambous et al., 2010) to analyze the content of the curriculum materials. Vertical analysis, also referred to as macro analysis by others (e.g., Li, Chen, & An, 2009), examines curriculum materials to identify what mathematics is taught at what grade levels. Thus, it was the primary method to answer the first research question. In contrast, horizontal analysis, also referred to as micro analysis by others (e.g., Li, Chen, & An, 2009), focuses the analysis on a particular mathematics topic and how it is treated in the curriculum materials. In other words, horizontal analysis may identify the progression of ideas through the curriculum materials.

RESULTS

The results of the content analysis will be presented according to the two research questions.

RQ 1: Which ideas are discussed in what grade level(s)?

Table 1 summarizes the topics related to ratio, rate and proportional relationship discussed in the Japanese COS. Ideas related to ratio, rate and proportional relationships are found primarily in two domains, Measurement and Quantitative Relationships. The Quantitative Relationships domain includes three sub-domains--ideas related to functions, expressions and equations, and data handling. The topics of two co-varying quantities, ratio, and proportional relationships are found in the Function sub-domain. Per unit quantity and speed are found in the Measurement domain.

Grades	Topics
4	Relationships of two co-varying quantities
5	Per unit quantity (comparison of two quantities from different measure spaces)
	Percentage
	Simple proportional relationships
6	Speed
	Ratio
	Direct and inverse proportional relationships
7	Direct and inverse proportional relationships

Table 1. Topics related to ratio, rate and proportional relationships in COS.

Table 2 lists the textbook series units and their mathematical content related to the topics identified in the COS. In the Grade 4 unit, students explore a variety of co-varying quantities, including proportional relationships. However, no explicit mention of proportional relationships occurs in the unit. The COS specifies that simple proportional relationships are to be discussed in Grade 5, but no unit focuses on that idea. Rather, the idea of proportional relationships is introduced in the unit on volume through the following problem:

> As shown on the right (accompanying figure is omitted), we are going to change the height of a cuboid from 1 cm to 2 cm, 3 cm, … without changing its length or width. Investigate how the volume changes. (Fujii & Iitaka, 2012, Grade 5, p. A20)

Students are expected to organize the results in a table showing various heights and corresponding volumes. Upon conclusion of this investigation, the textbook provides the definition of proportional relationship as follows:

> Suppose there are two quantities, □ and ○. If ○ becomes 2, 3, … times as much while □ becomes 2, 3, … times as much, we say that "○ is **proportional** to □." (Fujii & Iitaka, 2012, Grade 5, p. A20, emphasis original)

This one-page investigation is the only discussion of proportional relationships in Grade 5. However, this is consistent with how *Teaching Guide* defines a "simple" proportional relationship. *Teaching Guide* specifically states that "a simple case means students become aware that sometimes when one quantity becomes 2, 3, 4, … times as much, the other quantity will also become 2, 3, 4, … times as much" (translated by Author).

Grade	Title of unit	Mathematical content
4	How do quantities change?	Expressing the relationship of two co-varying quantities in equations using symbols.
5	Let's think about how to compare (1)	Arithmetic mean
		Per unit quantity
	Let's think about how to compare (2)	Percentage
6	Let's think about how to express proportions	Ratio
	Let's think about how to express speed	Speed
	Let's investigate proportional relationships	Direct proportional relationship
		Inverse proportional relationship
7	Direct and Inverse proportions	Direct proportional relationship
		Inverse proportional relationship

Table 2. Titles of units in the textbooks (Fujii & Iitaka, 2012) where topics related to ratio, rate and proportional relationships are discussed. The titles are listed in the order they appear in the textbooks.

RQ 2: How are the ideas introduced and developed?

Because of the space limitation, we will only share the results concerning proportional relationships. As noted above, proportional relationships are introduced in Grade 5 and further developed in Grades 6 and 7. Thus, we tried to answer the following questions: What are the emphases in each grade, and what new ideas are being introduced?

The idea of proportionality is introduced in Grade 5 as students investigate the relationship between the height and the volume of rectangular prisms with the same base. Students are asked when the height becomes 2, 3, … times as much, how the volume changes, and the definition of a proportional relationship is given as shown above. While investigating two co-varying quantities in Grade 4, the textbook asked several times how one quantity changes as the other quantity increases by 1 unit. Thus, investigating how a quantity changes when the other quantity becomes so many times as much is a new perspective introduced in Grade 5.

In Grade 6, students are given a table showing the number of minutes and the depth of water as water is being poured into a fish tank. The textbook explicitly states that the depth of water is proportional to the number of minutes. Students are then asked to find the quotients when the depth of water is divided by the amount of time to discover that the quotients of corresponding quantities are constant. Students then conclude that if y is proportional to x, their relationship can be expressed in an equation, y = (fixed number) × x. Later in the unit, this relationship is contrasted with inversely proportional relationships where the products of the two corresponding quantities are constant and their relationship can be expressed in an equation, y = (fixed number) ÷ x.

In Grade 5, students only considered what happens to the corresponding quantities when the other quantities become 2, 3, 4, … times as much; that is, only whole number multiples. In Grade 6, students discover that the same relationship holds even if the scale factors are positive decimal numbers or fractions. Students also graph proportional relationships and are expected to discover that graphs of proportional relationships will be straight lines that start on the bottom left corner of the graph paper.

In Grade 7, there seem to be two foci in the units of direct and inverse proportional relationships. The first emphasis is understanding the idea of functions, which is the topic of the first sub-unit. Students examine proportional relationships as a type of functional relationship. As students learn about functions, they are introduced to the idea of domain and range. The second emphasis in Grade 7 is to extend the range of the domain, the range and the constant of proportionality – these formal terms are introduced in Grade 7 – to the entire set of rational numbers. Students will then be able to graph proportional relationships on x-y coordinates, showing all four quadrants. They learn that the graph of a proportional relationship is a straight line through the origin.

DISCUSSION

The analysis shows that the Japanese curriculum treats ideas related to ratio, rate, and proportional relationships carefully and systematically. Moreover, beginning in elementary school, *Teaching Guide* clearly locates the study of proportional relationships within a broader study of functions. Although the idea of proportional relationships is discussed in four grade levels, there are new and mathematically more sophisticated expectations in every grade level.

The Japanese curriculum clearly distinguishes the multiplicative comparison of two quantities from different measure spaces from that of two quantities from the same measure space. Both types of comparisons are discussed in Grade 5, but the idea of per-unit quantity—that is, comparison of two quantities from different measure spaces—occurred in Grade 6 in the previous revision of the COS. Thus, the order of introduction of comparison types still seems to be an open question among Japanese mathematics educators.

LIMITATIONS AND FUTURE RESEARCH

One limitation of the current study is that only one textbook series was analysed. Although the grade level placement of a specific topic is defined by the COS, the way a particular series sequences topics within a grade level is left up to each publisher. Examining other series would surely deepen our understanding of the way the Japanese curriculum treats these important ideas. Also, *Teaching Guide* discusses the relationship between proportional relationships and operations of multiplication and division with decimal numbers and fractions. Those topics were not included in this study's analysis. Expanding the scope of analysis may give us additional insights about the treatment of proportional relationships in Japanese curriculum materials.

References

Charalambous, Y. C., Delaney, S., Hsu, H., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*, 117 – 151.

Fujii, T., & Iitaka, S. (2012). *Mathematics International*. Tokyo: Tokyo Shoseki.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Li, Y., Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: The case of fraction division. *ZDM Mathematics Education*, *41*, 809–826.

Tabata, T. (2010) Wariai, hi, hirei [Wariai, ratio and proportional relationships.] *Arithmetic Education, Journal of Japan Society of Mathematical Education*, *59*(6), 60–61.

MATHEMATICAL POWER IN THE MATHEMATICS IN CONTEXT CURRICULUM

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In the United States, Teaching Mathematics for Social Justice (TMSJ) has the potential to make studying advanced mathematics more accessible to students coming from low income and historically underrepresented ethnic groups. One way TMSJ has been conceptualized is by Gutstein's (2006) work on teaching students to "read" and "write" the world with mathematics. The purpose of this textbook analysis is to determine the nature of "mathematical power" within the ratio and rate sections in the "Mathematics in Context" curriculum. Results suggest that Gutstein's characterization of the mathematical power in Mathematics In Context is accurate and may be useful to teachers interested in teaching students to "read" and "write" the world with mathematics.

INTRODUCTION

From my prior experiences teaching in an urban school district, I gained a firsthand account of the struggles that students from ethnic minority or low socioeconomic status families face in learning mathematics. The phrase *teaching mathematics for social justice* (TMSJ) has brought a feeling of hope that future students coming from diverse and low-income households can be granted the same access to the mathematics community as their higher achieving peers. One conceptualization of TMSJ in a curricular context is Gutstein's (2006) work on *Reading and Writing the World With Mathematics (RWWM),* however, much of Gutstein's focus was on curricula resources that were not a part of his district-provided resources. Drawing from my personal experience struggling to meet the expectations of my district while enacting what I believed to effective instruction, it is my personal goal for this study to better understand how curricula may allow for *mathematical power* (MP) while at the same time developing an essential understanding of the same mathematical concepts as a traditional curriculum. The purpose of this study is to determine what MP looks like in the *written curriculum* of the *standards-based* (Stein, Remillard, & Smith, 2007) middle school textbook Mathematics in Context (Holt, Rinehart, & Winston, 2006) (MiC). This analysis could help to validate the viability of using this curriculum and my conceptualization of MP to support a Reading and Writing the World with Mathematics (RWWM) (Gutstein, 2016) pedagogy.

FRAMEWORKS

In Gutstein (2006), the author tells of his experience as a teacher researcher in a diverse, urban middle school. He approached his practice during this project with a focus on connecting his curriculum, MiC, with social issues in or near the community in which

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he taught. Although, much of his social justice lessons came in the form of special projects, one of Gutstein's conclusions was that:

> "MiC by itself does not prepare students to read and write their worlds with mathematics" (p. 104) but it is "so strong in developing mathematical power in my students that it actually provided me a certain amount of 'pedagogical space' ... [to] engage my students in using mathematics to investigate racism, wealth inequality, costs of war preparations, gentrification, and other issues" (p. 107).

Since MiC's connections to social justice are dependent on its ability to instill MP in students, I chose to build my conception of MP from both Gutstein's and the National Council of Teachers of Mathematics' (NCTM) Curriculum and Evaluation Standards for School Mathematics (1989). NCTM defines MP as:

> "an individual's abilities to explore conjecture, and reason logically, as well as the ability to use a variety of mathematical methods to solve nonroutine problems, this notion is based on the recognition that mathematics is more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication, and notions of context ... mathematical power involves the development of personal self-confidence" (NCTM, 1989, p. 5).

There are some slight nuances in the way that Gutstein (2006) describes students' engagement in MP:

> "students confidently engage in complex mathematical tasks, draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress ... are flexible and resourceful problem solvers ... work productively and reflectively ... communicate their ideas and results effectively ... value mathematics and engage actively in learning it." (p. 6)

My conception of MP in a textbook setting is more closely aligned with Gutstein's definition with a more explicit emphasis on critique and connection to the real world. My idea of MP in textbooks could be summed up as the development of students' competencies in communicating about complex problems in both a mathematical and real-world context. This includes encouraging the use of multiple representations and multiple problem-solving strategies to generate multiple ways of thinking about the problems at hand. This is done through communication of the logical reasoning behind their work and critique of others' ideas related to the context. This definition accurately and concisely captures both Gutstein (2006) and NCTM's (1989) ideas while also explicitly adding in the elements of connection to real-world contexts and critique of others. This was done because I believe these elements add a more human component to the idea and encourage an understanding of the problems one may be faced with in their community.

The second important aspect of determining the nature of MP in MiC was the classification of the mathematical content evident in the textbook sections covering Ratios and Rates. To do this, I call on Lobato et al.'s (2010) description of ten Essential Understandings (EUs) that students should develop as they learn about ratios, rates, and proportions. Of the ten EUs, the first five are most closely related to the study of ratios and rates. Thus, the analysis for this study will be centered around these five EUs. The decision to include Lobato et al.'s (2010) work was twofold: (1) stemming from my conception of mathematical power, I felt it was necessary to include a classification of what mathematical contexts students were reasoning about and (2) the authors offer a concise and manageable number of ideas to try to operationalize. The choice to limit the number of EUs considered was made because it is not the intent of this paper to gain full access to students' understandings of ratios, but merely see how they are initially introduced to ratios and rates. A more thorough description of the EUs can be found in Lobato et al. (2010), but my choice to look for "explicit" EU content was used as an attempt to make these observations more objective. By explicit, I mean that the item contains text, a diagram, table, or graph that implies utilization for completing the task. The indicators also use the word "representation." I considered a representation to be a table, graph, diagram, text, picture, or any other visual aid that conveys mathematical ideas.

METHODS

In considering whether a problem from the textbook exemplified MP, I considered the following Mathematical Power Questions (MP-Q):

MP-Q1: Are students are presented with or encouraged to use multiple ways to solve a problem?

MP-Q2: Are students presented with or required to find multiple correct answers?

MP-Q3: Are students presented with or required to critique others' ideas?

MP-Q4: Are students presented with or asked to provide an explanation or defense of their ideas?

One can see that the first two of these MP-Q's are closely aligned with both NCTM and Gutstein's definition of MP whereas the last two are adapted from Gutstein's suggestion that MP necessitates students communicating their thoughts. An example of a problem that I considered to be a good indication of MP is given in figure 1. This problem was coded as meeting both MP-Q3 and MP-Q4 because of how it asks students to comment on the judges' decision as well and how is asks for students to "explain your reasoning."

When timekeepers used hand-held stopwatches, it was very difficult to rank evenly matched competitors. At the 1960 Olympic games in Rome, Australia's John Devitt and America's Lance Larson finished neck-and-neck in the final of the 100-m freestyle swimming events. All three timekeepers for Devitt's lane clocked him at 55.2 sec. Larson was clocked at 55.0, 55.1, and 55.1 sec. The judges placed Devitt as the winner. The official time for both swimmers was recorded as 55.2 sec.

7. a. Is this fair? Explain your reasoning using your knowledge about reaction time.

b. Suppose Larson swam 100 m in 55.2 sec and Devitt finished 0.1 sec before Larson. What is Larson's distance (in cm) from the wall when Devitt finished the race? Would this have been visible?

Figure 1. Item 3-1A-C7 from MiC (2006)

In order to categorize the mathematical content within MiC the first five EUs were operationalized into the following Essential Understanding Questions (EU-Q):

Essential Understanding Questions	
EU-Q1	Are students provided with an explicit representation involving the relationship between both quantities?
EU-Q2a	Are students provided with an explicit representation of the multiplicative change in two quantities?
EU-Q2b	Are students provided with an explicit representation of an iterative view of a composed unit?
EU-Q3a	Are students provided with an explicit representation of the effect of changing one quantity in relation to the other quantity?
EU-Q3b	Are students provided with an explicit representation of the "attribute of interest" in relation to the two quantities?
EU-Q4a	Are students provided with an explicit representation linking ratios and fractions?
EU-Q4b	Are students provided with an explicit representation of "part-part" comparison?
EU-Q4c	Are students provided with an explicit representation considering fractions as ratios?
EU-Q5a	Are students provided with an explicit representation demonstrating division in relation to ratios?
EU-Q5b	Are students provided with an explicit representation demonstrating unit rates?

Figure 2. Essential Understanding Questions

I analyzed a total of 39 items sampled from seven sections of the three MiC textbooks for the presence of both EU indicators and MP indicators. The sampling was limited to one section from the Level 1 textbook, five sections from the Level 2 textbook, and one section from the Level 3 textbook. The sections in the Level 1 and Level 3 textbook were chosen because they were the students' first interaction with ratios and rates within the textbook. The sections in the Level 2 textbook were chosen because they all fell under the textbook unit titled "Ratios and Rates," and the Level 3 section was focused on development of proportional reasoning, but begins with ratios and rates. Within each chosen section, only the Summary, Check Your Work, and For Further Reflection components were coded for the presence of indicators. Within each section, each problem under a subheading was considered as the unit of analysis. Throughout the paper I refer to these as "items." Since the coding was completed by only one researcher, it was reiterated on three separate occasions to check for consistency. Since the Check Your Work items included answers, my first round of coding was compared with the suggested answers to verify the authors' intent for the item.

RESULTS AND COMMENTARY

From section to section there was a relatively consistent occurrence of EU indicators. This was determined by analyzing the number of indicators per item in each section. This was calculated for each section to determine if any single section contained a drastically disproportionate number of indicators. Of the 39 items, 37 of them showed evidence of at least one EU indicator and 19 items showed evidence of at least one MP indicator. The EU-Q5b, which concerned the use of unit rates as a method of problem solving, was observed in approximately 64 percent of the items. This may suggest that this is a preferred method of the MiC authors. This preference may contribute to a dependence on using unit rates to solve problems which would make for less flexibility and hence less MP. I also found that MiC is lacking when it comes to comparing and contrasting ratios with fractions. Considering that this curriculum's developers claim that "student interaction is essential" (Encyclopædia Britannica, Inc., 2018), I do find it interesting that the only instance where the distinction between fractions and ratios is made was in a Summary component. This may suggest that this EU has been overlooked by the authors of MiC. With respect to the MP-Qs, I found that the most common way of promoting mathematical power in MiC was shown by asking students to provide an explanation or defense of their ideas (MP-Q4). I was not surprised by this considering it is quite common in standards-based textbooks to see questions with directions instructing students to "explain" or to "provide a reason for your answer." An interesting to follow up to this study may require a measure of "pedagogical space" to determine if the content of certain textbook sections inherently allows for MP and hence "pedagogical space" to incorporate topics of social justice. One context of interest in the Level 2 edition concerned the relationship between pollution and the relative number of vehicles on the road in a city. This was also of interest because it seems to be a good opportunity to bring resources outside of the written curriculum to further the mathematical conversation. It may be the case that by introducing students to this context and equipping them with strategies to think about climate change that the authors are allowing teachers to have "pedagogical space" for encouraging students to play a more active role in expressing their opinions in national conversations. Another item in particular that stood out to me as promoting mathematical power is shown in Figure 2 of the methods section. I found that this item placed the students in a position to critique the ideas of the judges in the context above. I found this interesting because this seems to be an instance where students are encouraged to speak out against a representation of power in society which lends itself to becoming more politically active. Though the prevalence of MiC in U.S. middle schools is indeterminable at this point, it is my hope that this study could bring about a practical way of assessing written curricula (Stein, Remillard, & Smith, 2007) for opportunities to incorporate TMSJ while still trying to meet the expectations of their district.

References

Encyclopædia Britannica, Inc. (2018). *Philosophy and goals of MiC. Teacher implementation guide*. Retrieved from http://mathincontext.eb.com/resources/files/implementation/Philosophy_and_Goals_of_MiC.pdf

Gutstein, E. (2006). *Reading and writing the world with mathematics*. New York: Routledge.

Gutstein, E. (2016). "Our issues, our people—math as our weapon": Critical mathematics in a chicago neighborhood high school. *Journal for Research in Mathematics Education*, *47*(5), 454. doi:10.5951/jresematheduc.47.5.0454

Holt, R., & Holt, W. (2006). *Mathematics in context: Level 1*. Chicago, Illinois: Encyclopaedia Britannica.

Holt, R., & Holt, W. (2006). *Mathematics in context: Level 2*. Chicago, Illinois: Encyclopaedia Britannica.

Holt, R., & Holt, W. (2006). *Mathematics in context: Level 3*. Chicago, Illinois: Encyclopaedia Britannica.

Lobato, J., & Ellis, A. B. (2010). *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 319–370). Charlotte, NC: Information Age Publishing.

Part V.

Poster

THE PRESENCE OF GEOMETRY IN FRACTION TASKS: AN ANALYSIS OF TEXTBOOKS

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INTRODUCTION

Brazil has the National Textbook Program (PNDL), which was created in 1985 with the purpose of evaluating the quality and facilitating the distribution of these materials (Pitombeira, 2008). Currently, textbooks are freely distributed for students in the public school system. Publishers must submit their textbooks to be evaluated for content, technical and physical aspects. The approved collections make up the Textbook Guide, a catalogue that contains the summary of the collections and can be accessed for the teachers, who choose the book they want to use.

As a mediator between the official curricula and the classroom and for structuring the teachers' pedagogical work, the textbook becomes an important source for the teaching and learning process (Amaral, Ribeiro, & Godoy, 2014; Batista, 2002). For this reason, it is relevant to analyse the content that has been presented in these materials.

METHODOLOGY

In this paper, we show the results of a research, still in development, that aims to investigate: how Geometry is present in other branches of Mathematics in some textbooks? Results have shown us that Geometry is present in contents such as irrational numbers, algebraic expression, systems of equations, etc. We make a cut here, focusing on the fractions tasks in textbooks destined to the last years of Elementary education. In total, there are ten collections approved to compose the catalogue of PNLD 2017, however, the textbooks are not tradable and therefore difficult to access. Thus, we are looking at two sixth-grade books of different collections (because they contain most of the fraction content). First, we selected the chapters dealing with the notions of fractions, and then separated the tasks in which Geometry appears. Table 1 shows the total number of fractional tasks and the number of fractional tasks with Geometry.

Books	Total of tasks	Tasks with Geometry
Book 1	504	72
Book 2	288	37
Book 3	232	27
Total	1024	136

Table 1. Distribution of fractions tasks and fractions tasks with Geometry

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 367–368). Paderborn: Universitätsbibliothek Paderborn.

These exercises were studied based on Arcavi (2003), in which the visual representations are understood as (a) a support to prove results; (b) a way to correct wrong intuitions; (c) an option to recover conceptual foundations.

RESULTS

Geometry appears in tasks that work the correspondence between fractional numbers and points of the numerical line and in the fraction representation through the decomposition of geometric figures or solids. In these terms, the representation of a rational number, purely symbolic, by a geometric figure can help the student understand what the whole and its parts are. In addition to offering the possibility of working equivalence between areas of flat figures. Some tasks all at once approach fractions and percentages using geometric representation. In this case, the teacher can show the students the relationships between fraction, percentage and reason.

Regarding the representation of fractions through the decomposition of geometrical solids, students are being introduced to basic concepts of volume that will be deepened in the following years. For now, an intuitive idea of what volume is being formed.

The correspondence between fractional numbers and points in the numerical line reinforces once again that fractions are rational numbers. The Book 1 encourages students to think of addition and subtraction operations of fractions with the same denominator, as "jumps" in the number line. Thus, the visual representation of arithmetic operations is used to justify why to add/subtract only the numerator and not the denominator.

References

Amaral, R. B., Ribeiro, M., & Godoy, J. S. (2014). Choosing textbooks without looking at the textbooks: The role of the other’s interpretation. In K. Jones, C. Bokhove, G. Howson, & L. Fan (Eds.), *Proceedings of the International Conference of Textbook Research and Development (ICMT-2014).* (pp. 153–158). Southampton: University of Southampton.

Arcavi, A. (2003). The role of visual representation in the learning of mathematics. In: *Educational Studies in Mathematics, 52*, 215–241.

Batista, A. A. G. (2002). *Recomendações para uma política pública de livros didáticos.*

Pitombeira, J. B. (2008). Políticas públicas e o livro didático de Matemática. *Boletim de Educação Matemática, 21*, 1–11).

THE USE OF A DIGITAL TEXTBOOK WITH INTEGRATED DIGITAL TOOLS

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THEORATICAL BACKGROUND

The PISA study reports „there is little solid evidence that greater computer use among students leads to better scores in mathematics“ (OECD 2015, p. 145). Furthermore, Drijvers et al. (2016) show that only small effects can be found in quantitative studies that research the effects of technology on students' outcomes. This is why the question arises how students can benefit from technology and digital tools in the classroom.

Since 2019, the project KomNetMath examines the use of a regularly used digital textbook with integrated digital tools. Because of the important role of textbooks, it can be presumed that textbooks affect didactical situations and the teaching organisation. Besides that, the TIMS study shows that the textbook is an influential factor that translates policy into pedagogy (Valverde et al., 2002). Valverde et al. (2002) argue that the structure of the textbook makes an impact on students‘ learning effectiveness: "Textbook form and structure advance a distinct pedagogical model" (Valverde et al., 2012, p. 54). The structure of a digital textbook differs from a traditional textbook in its dynamic elements (Pohl & Schacht, 2017). With regard to the progress of mathematical competencies and math achievement, there is still a high need for research on digital mathematics textbooks. Most studies in the field of textbook research refer to traditional (printed) textbooks and investigate the students' and teachers' use of mathematical textbooks (e.g., Johansson 2006). The textbook is regarded as an artefact that is used by teachers and students directly or indirectly (Rezat & Sträßer, 2012; Johansson, 2006). For example, using the textbook as a workbook in the classroom would be a direct use. In contrast, using it for lesson planning would be an indirect use of the textbook. Using a digital textbook with dynamic elements in this project, we examine how the use of a digital textbook affects learning as well as math achievement and the following research questions come up:

RESEARCH QUESTIONS

1) How do students develop mathematical competencies when using a digital textbook compared to students who use traditional (printed) textbooks? 2) How do teachers and students use the digital textbook in teaching and learning? 3) How do the students' beliefs towards the use of digital textbooks change when using a digital textbook?

METHOD

In the project KomNetMath, teachers and students regularly use the digital textbook in grade 10 and 11 at German schools since 2019. The control group will get the same

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 369–370). Paderborn: Universitätsbibliothek Paderborn.

learning environment, but this group will instead work with only printed textbooks and materials. The teachers record the use of the textbook by filling out lesson reports every lesson. Teachers have to report on how the textbook is used and didactical decisions they carried out. For this reason, the lesson reports include items about the classroom organisation and the direct or indirect use of the textbook. The investigation of the students' competence increase will be conducted to a topic out of the curriculum in consecutive math lessons. We select a topic for which the digital textbook offers interactive and multimedia elements as well digital tools. In a pre- and post-test design, the mathematical competencies will be surveyed. We will focus on students' math achievement and in the pre-test the students' prior knowledge will be measured. Furthermore, the change of student's beliefs towards the use of digital textbooks when using one over an extended period is a research interest of the KomNetMath project. For that reason, the students will get an identical questionnaire at several measurement points. Therefore, the project examines affective as well as cognitive aspects of a digital textbook use. As support, the teachers participate in further trainings, whereby they get to know the handling and opportunities of the digital textbook and its integrated tools. First results of the lesson reports and questionnaires will be presented at the conference.

References

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Cham: Springer.

Johansson, M. (2006). *Teaching mathematics with textbooks: A classroom and curricular perspective* (Doctoral dissertation, Luleå tekniska universitet). Retrieved from http://urn.kb.se/resolve?urn= urn:nbn:se:ltu:diva-25802

OECD. (2015). *Students, computers and learning: Making the connection*. PISA, OECD Publishing, Paris, https://doi.org/10.1787/9789264239555-en.

Pohl, M., & Schacht, F. (2017). Digital mathematics textbooks: Analyzing structure of student uses. In G. Aldon & J. Trgalová (Eds.), *Proceedings of the 13th International Conference on Technology in Mathematics Teaching / ICTMT 2017 3rd to 6th July, 2017, Lyon, France* (pp. 453–456). France.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM Mathematics Education, 44*(5), 641–651. https://doi.org/10.1007/s11858-012-0448-4

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht: Kluwer.

CURRENT STATUS AND ISSUES OF DIGITAL TEACHING MATERIALS IN JAPAN

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JAPANESE DIGITAL DEVICES AND DIGITAL TEXTBOOKS

Because there are many classes taught simultaneously in elementary schools in Japan, basic digital equipment is used to enlarge and present the digital textbooks on electronic blackboards or projectors. Digital textbooks are based on paper textbooks and are divided for teacher and children. Until now only textbooks for teachers were used, however, after 2020, digital school textbooks for children will also be introduced.

DIGITAL TEACHING MATERIALS IN MATHEMATICS EDUCATION

In Japan, personal computers began to be used in the latter half of the 1970s, and studies on its use in mathematics education began. Initially MS-DOS was used as basic software and BASIC was used to make teaching materials. After that CAI was introduced and changed to PC with MS-Windows since about 2010. Educational software came to be sold in the age of Windows, although the teacher was making all the programs in the era when MS - DOS was in use. However, the teaching materials being sold are not exactly the same as paper textbooks, so it was not easy to use. As a result, mathematics lessons using digital teaching materials are not very popular. On the contrary, devices that can easily expand and project textbooks like real projectors are spreading.

Currently, research on digital teaching materials, not digitized textbooks, is actively conducted, mainly by elementary school teachers and researchers of educational engineering. Also, the digital teaching materials have two main use purposes. One is the digital teaching materials according to the content of paper textbooks. Basically, it is used for supplementing the content of textbooks. The second is the teaching materials for learning free content to develop mathematical abilities without complying with the contents of textbooks.

RESEARCH SITUATION OF DIGITAL TEACHING MATERIALS IN MATHEMATICS EDUCATION

Research on digital teaching materials in mathematics education within about 10 years is summarized in the following four contents.

1) Lesson design / Class design

2) What kind of ability develops using digital teaching materials

3) What kind of scene / timing should you use with digital materials

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 371–372). Paderborn: Universitätsbibliothek Paderborn.

4) Distinction between commercial software and self-made teaching materials

CONCLUSION

Regarding the timing and scenes of utilization in the class, it is being studied which way of using PCs can be considered in each learning scene while maintaining the teaching process which is widely adopted in mathematics departments.

Regarding the usage of commercial software / self-made software, although it is possible to create more effective software by clarifying the purpose and aims of the lesson, there is a possibility that it may easily use inadequate software. It is important to design lessons in combination with the scenes used and the skills that they want to extend, as their functions and properties are clarified. Also, when creating teaching materials, it is also important to clarify the roles and intentions that the software has to achieve as much as possible.

Regarding class design and lesson design, there are not many lesson processes that can be a used as a model, so continued research is necessary in the future.

Research on the possibility of whether technology can be used for classes is still very important. For example, topics such as imaging of concepts, deepening of mathematical thinking ability, simulation included in interactive functions, modelling, which are considered to be important as subjects of research in mathematics education, are hardly addressed.

In addition, the four viewpoints that were organized are mostly mixed studies, and research aimed at dividing one by one is awaited. There are many problems surrounding the use of PCs. In the future, it is necessary to start research from the viewpoint that the use of PCs in mathematics education is effective, and to accelerate research from a micro viewpoint.

References

Horita, T., & Kihara, T. (2008). Current status and issues of ICT use in teaching to improve scholastic abilities in Japan. *Japan Journal of Educational Technology*, *32*(3), 253–263.

Matsubara, S. et. al (2014). Current situation and challenges in use of ICT for education: The ICT education research project by Toyo University Institute of Social Sciences: First Report. *The Journal of Contemporary Social Sciences*, (12), 61–72.

Suzuki, T. et. al (2015). Practice and evaluation of classes utilizing tablet devices in the first grade of primary school. *IPSJ Transactions on Computers and Education*, *1*(4), 21–3.

Hakozaki-Lee, H. (2015). A study on the feature of “teaching action” in utilizing digital textbook. *Research Bulletin, Tōin University of Yokohama*, (32), 37–42.

LIMIT OF TEXTBOOK IN CONCEPTUAL UNDERSTANDING

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BACKGROUND TO STUDY

Textbooks clearly play an important role in the teaching and learning of mathematics. Also, textbooks impact both what and how teachers teach, as well as what and how learners learn (Herbel-Eisenmann, 2007). In Japan, the government reviews the descriptions of textbooks, and the textbook authors have to adapt to the conditions of the curriculum. However, the examples used in textbooks are left to the decisions of its authors. Rowland (2008) writes "choices of examples and their sequencing are neither trivial nor arbitrary". Therefore, the examples authors decide on are important. However, if textbooks are not used, they are not useful. Bokhove and Jones (2014) mention, the allusion of 'over-reliance' on textbooks might have negative connotations and may have contributed to the relatively low usage of textbooks in mathematics classrooms in England. This shows that it is meaningless to force the use of textbooks. Thus, the situation must be improved to increase usage rates. Dietiker (2014) concluded that curricular reform should be recast as changes to previous stories. Authors should change textbooks according to curricular reform. Nevertheless, universal descriptions also exist. Vermeulen (2016) argues that, in all cases there has been development in sophistication of mathematical ideas. Mathematical ideas are indispensable in the improvement of textbooks for conceptual understanding. In this paper, our aim is to consider a solution based on the universal mathematical idea in Japanese mathematics textbooks.

METHODOLOGY

Rezat (2006) offers a model for analysing textbook use based on activity theory. His model adopts the three dimensional shape of a tetrahedron. Costa and Matos (2014) discusses the influence of the use of textbooks in elementary school teachers' practices when a new structure was created in the Portuguese educational system in 1968. They used Rezat’s model in their study.

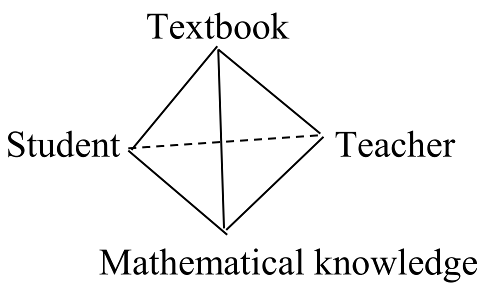

Figure 1. Rezat’s (2006) model

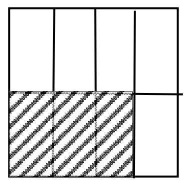

Figure 2. Figure of 1/2×3/4.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 373–374). Paderborn: Universitätsbibliothek Paderborn.

DATA

We consider the conceptual understanding in the multiplication of fractions. Teachers teach fractions in the sixth grade of elementary school. The conclusion is to apply numerators and denominators with each other. It is merely a procedural understanding. In order to make this a conceptual understanding, the area of the square is shown in the textbook. For example, the right figure illustrates $1/2 \times 3/4$.

DISCUSSION

Japanese fifth graders in elementary schools learn decimal $\times$ decimal. The teachers urge conceptual understanding based on the number line. However, those teachers do not use this number line in this scene. Thus, it is not used in textbooks. It can be expected that conceptual understanding is urged to examining the area of the square on the other hand. The relation of Textbook and Student, Teacher, and Mathematical knowledge is as shown in Rezat's model, by analysing lessons. Therefore, in the multiplication of decimals and fractions, it is concluded that the method of urging conceptual understanding is not related.

References

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the “voice” of a mathematics textbook. *Journal for Research in Mathematics Education, 38*(4), 344–369.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics, 69*(2), 149–163.

Bokhove. C., & Jones. K. (2014). Mathematics textbook use in England: Mining ofsted reports for views on textbooks. In Jones, K., Bokhove, C., Howson, G., & Fan, L. (Eds.), *Proceedings of the International Conference on Mathematics Textbook Research and Development.* (pp. 159–166). Southampton, UK: ICMT.

Dietiker, L. (2014). Telling new stories: Reconceptualizing textbook reform in mathematics. In Jones, K., Bokhove, C., Howson, G., & Fan, L. (Eds.), *Proceedings of the International Conference on Mathematics Textbook Research and Development.* (pp. 185–190). Southampton, UK: ICMT.

Vermeulen, N. (2016). Developing algebraic thinking: The case of South African grade 4 textbooks. In Csikos, C., Rausch, A., & Szitanyi, J. (Eds.), *Proceedings of the 41th Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 4, pp. 339–346). Szeged, Hungary: PME.

Rezat, S. (2006). A model of textbook use. In Novotná. J., Moraová. H., Kráthá. M., & Stehliková. N. (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 4, pp. 409–416). Prague, Czech: PME.

THE ANALYSIS OF THE DIAGRAMS IN THE TEXTBOOK OF ARITHMETIC

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THE PURPOSE AND METHODS OF THIS STUDY

It is useful and meaningful to use diagrams for making sense of calculation and solving word problems in elementary school. However, it is difficult for pupils to interpret and use the diagrams. In response to this situation in Japan, we need to reconsider using diagrams in teaching/learning calculations for the development of number concepts and meaning of calculation from the cross-sectional perspectives. Therefore, the purpose of this study is to examine and consider features and classification of diagrams in textbooks used for teaching/learning calculations in elementary school. As a theoretical framework, based on the theory of mathematical representation by Nakahara (1995) and the process of solving word problems by Mayer (1992), we analyse and consider diagrams used in textbooks. In this paper, we cover teaching/learning addition and subtraction of integers in Japanese textbooks.

THEORETICAL FRAMEWORK: MATHEMATICAL EXPRESSIONS AND A PROCESS OF SOLVING WORD PROBLEMS

First, Nakahara (1995) showed illustrative representations used in mathematical education. The classification of them by Nakahara is as follows: the reality diagram, the mathematical scene diagram, the procedure diagram, the structure diagram, the concept diagram, the rule and relation diagram, the graph diagram and the geometrical figure. Next, Mayer (1992) showed the process of solving word problems. It has two steps, "Problem representation" and "Problem solving". Each of the steps is additionally divided into two processes. "Problem representation" is divided into "Problem translation" and "Problem integration". "Problem solving" is divided into "Solution planning and monitoring" and "Solution execution". Mayer stated that when we solved word problems, we went back and forth between the steps. We also consider that the process of solving word problems is useful framework because it matches a flow of teaching/leaning calculations. In the textbook, we consider there is the flow of the teaching/learning calculations as shown in the middle of Figure 1. Also, in Figure 1, we indicate the relation between this flow and the process of solving word problems by Mayer.

CONCLUSION 1: CLASSIFICATION OF DIAGRAMS IN THE TEXTBOOKS FOR TEACHING/LEARNING CALCULATIONS

Reviewing Nakahara's study, we consider there are some issues about his classification; some diagrams classified in the mathematical scene diagram, the

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 375–376). Paderborn: Universitätsbibliothek Paderborn.

procedure diagram and the structure diagram should be in a different group of the diagram classification. Therefore, we organize the diagram classification and reclassify the reality diagrams, the mathematical scene diagrams and the structure diagrams appropriately. Furthermore, we arrange them with the flow of teaching/learning calculations in the textbook.

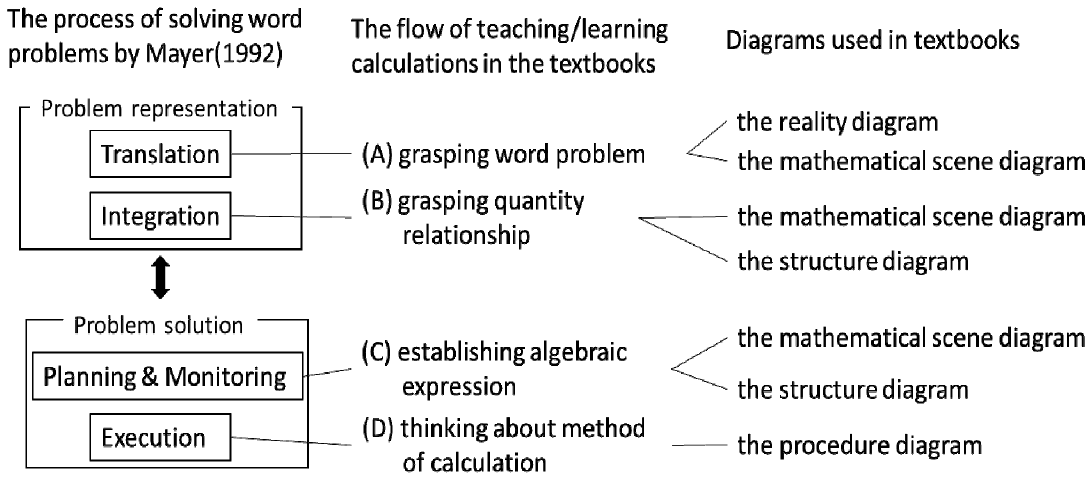

Figure1: The flow of teaching/learning calculations in the textbook and the diagrams

First, in (A), the reality diagrams are used. This is considered to make it easier to recall the problem scene presented in the sentence. Second, in (A), (B) and (C), the mathematical scene diagrams are used. Moreover, in (B) and (C), it is considered that the structure diagrams are used to clarify the kind of the word problem and connect it to the standpoint. Finally, in (D), the procedure diagrams are used.

CONCLUSION 2: DIFFERENCES OF DIAGRAMS ACCORDING TO DEVELOPMENT OF PUPILS

First, the reality diagrams become rough as the grade level goes up. For example, in the first grade, there are many diagrams mirroring the problem sentence, but as the grade level goes up, the number of diagrams only drawing the scene of the problem increases. Second, the mathematical scene diagrams are often used in the first grade, but they are rarely used as the grade level goes up. Next, the structure diagrams have different abstraction degrees for each grade. In the first and second grade, there are many figures in which the subjects appearing in sentences are concretely expressed. On the other hand, in the end of the first grade, figures expressing the object by signs are used. Then, they use the tape diagrams in the second grade and line segments in the third grade.

References

Mayer, R. E. (1992). *Thinking, problem solving, cognition*, New York: W. H. Freeman and Company.

Nakahara, T. (1995). *The Study of constructive approach in mathematics education.* Japan: Seibunsha Publishing. (in Japanese)

A STUDY ON THE EDUCATIONAL IDEOLOGY OF MODERNIZATION IN THE CASE OF JAPANESE TEXTBOOK

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INTRODUCTION

The purpose of this study is to describe an ideology of era of modernization in mathematics education through our theoretical framework. According to Chevallard (2015), our current ideology is paradigm ‘of visiting monuments’, yet a paradigm ‘of questioning the world’ should replace it. In the paradigm ‘of visiting monuments’, the purpose of education is studying “grand systems”, “great man” supposed to have authored. Contrary to that, in the paradigm ‘of questioning the world’, purpose of education is to foster students’ attitude called ‘Herbartian’, meaning “…towards yet unanswered questions and unsolved problems, which is normally the scientist’s attitude in his field of research and should become the citizen’s in every domain of activity (Chevallard, 2015, p. 178)”.

THEORETICAL FRAMEWORK

Here, we focus on the term ‘paradigm’ which is proposed by Kuhn (1970) because it was critiqued by Lakatos’s (1978) notion of ‘research program’. The term ‘paradigm’ can not describe people’s efforts, as Kuhn said of conversion experience, and there is no rational standard. To describe people’s efforts and to make rational standard clear, Lakatos (1978) constructed ‘research program’. It analyses that a series of theories have ‘hard core’, which is consistently kept, and ‘protective belt’, which is part that modulated to correspond counterexamples. When the protective belt can’t correspond counterexample, the counter research program reach to hard core and replace it.

Because of necessity of considering belief in education, we can’t apply it to education. Then based on it, we constructed our hypothetical framework ‘educational research program’, which consisting inseparatable three levels of ‘hard core’ and ‘protective belt’, namely ‘rational level’, ‘belief level’, and ‘society level’; ‘rational level’ is identical with Lakatos’s (1978) research program. Using this theoretical framework, we can consider what happens in the single paradigm, for example, what is an ideology of theory and what did a theory do to the paradigm ‘of visiting monuments’.

METHODOLOGY

Our methodology is philosophical consideration for mathematics textbook because it reflects the social and ideological conditions and constraints in era of modernization. The reason for paying attention to modernization is that, it is one of most important

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 377–378). Paderborn: Universitätsbibliothek Paderborn.

educational movements and, in Japan, the curriculum clearly stated that the purpose is to foster students' attitude for first time.

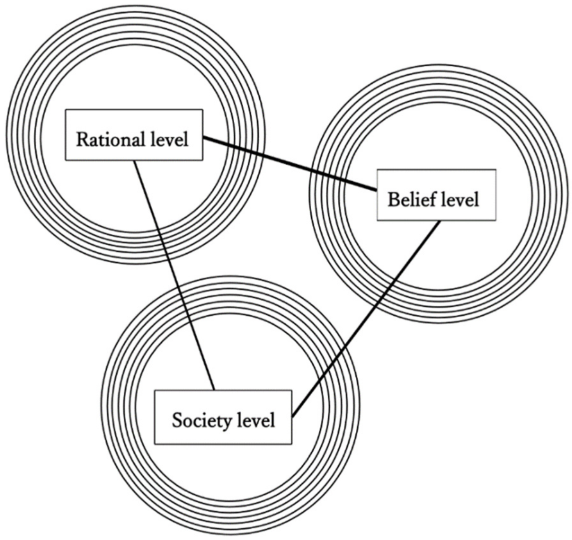

Figure 1. Figure of educational research program

RESULT

As a result, we can describe that modernization attacked or, in other words "falsified (Lakatos, 1978)", only the protective belt of the 'belief level' and the 'society level'; for example, it changed the contents and arrangement of mathematical knowledge in curriculum. This attack failed to reach the hard core in each. Thus, modernization cannot change the hard core so it become new protective belt for monumentalism.

In this paper, we only described only phenomena on modernization. Our future tasks are to answer question of why the modernization attacked only the protective belt of the 'belief level' and 'society level', and why failed the attack to reach the hard core of the 'rational level'.

References

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *The Proceedings of the 12th In International Congress on Mathematics Education* (pp.173–187). Cham: Springer.

Kuhn, T. (1970). *The structure of scientific revolutions* (Second Edition, Enlarged). Chicago: The University of Chicago Press

Lakatos, I. (1978). *The methodology of scientific research programs.* Cambridge: Cambridge University Press

Syoda, K., Hashimoto, J., Kurita, M., Kikuchi, H., Sugioka, S., … Nakamura, K., (1971). *Sugaku 2 (Grade 8)*. Osaka: Keirinkan. (in Japanese)

Takahashi, K. (1999). What is scientific research program for history of science researchers. In *Bulletin of Graduate School of Social and Cultural Studies* (vol. 5, pp. 27–39). Japan: Kyushu University. (in Japanese)

GEOMETRICAL PROPERTIES AS ASSUMPTIONS IN PROOFS IN JAPANESE JUNIOR HIGH MATHEMATICS TEXTBOOKS

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INTRODUCTION

Given the international evidence that students have trouble in understanding proofs (e.g., Hanna & de Villiers, 2012), the appropriateness of the content of textbooks is one of crucial factors (Fan et al., 2018). Here, we focus on the introduction of deductive proof in Japanese junior high school mathematics textbooks. In major textbooks authorized by the Japanese Ministry of Education, a deductive proof is defined as *an explanation of a statement based on properties that are already known to be true*. If these 'already known to be true statements' (i.e. assumptions) are treated ambiguously in the textbooks, then this might cause some of the difficulties in the teaching and learning of proofs. Hence our research question is: in Japanese junior high school mathematics textbooks, what is the relationship between assumptions prescribed in the definition of a proof, and the properties that appear as statements to be proved?

ANALYTIC FRAMEWORK AND METHOD

In order to examine the relationships, we define *correspondence* as the consistency between facts for students and properties, and *coherence* as the deductive consistency between properties. Using two major authorized textbooks, Tokyosyoseki (T) and Keirinkan (K), with over 70% of national share of classroom use, we examined all proofs in the 8th grade textbooks where a mathematical proof was introduced. In doing so, we analysed the relationships between assumptions prescribed in the definition of a proof and properties used in the proofs in the textbooks.

RESULTS

Through our analysis, we found that although the assumptions in these two textbooks were defined as *properties that are known to be correct*, some of this was without mentioning the methods of verifying the correctness (e.g. by demonstration, deduction etc.). The number of assumptions which come under the definition of a proof before introducing this definition (Group A) was 13(T) and 12(K) respectively. The number of properties whose correctness was verified by deductive proofs after introducing the definition of proof (Group B) was 12(T) and 6(K). By divided Group A into the two groups, Group $A \cap B$ and Group $\{A\text{-}A \cap B\}$, we found that properties of Group $A \cap B$ were proved after introducing the proof definition, although the all properties of Group A were already 'stamped' as being already known as correct. As such, by distinguishing the two criteria of purposeful correctness, 'correspondence' and 'coherency', we found that a premediated distinction was being made in the textbooks

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 379–380). Paderborn: Universitätsbibliothek Paderborn.

as to properties of Group A that were proved and those which were not. For instance, in both authorized textbooks while the conditions of congruent triangles is used as an assumption of proofs without being proved (Group {A-A∩B}), the property of the base angles of isosceles triangle is proved (Group A∩B), Although many properties belonged to Group {B-A∩B}, we found that 5(T) and 2(K) properties belonged to Group A∩B (e.g. the property of the base angles of isosceles triangle), and 8(T) and 12(K) properties belong to Group {A-A∩B}(e.g. the conditions of congruent triangles). 'correspondence' + 'coherency').

Textbook	Assumptions which come under the proof definition (Group A)	Properties whose correctness was verified by deductive proofs (Group B)	Group A∧B
T	13	12	5
K	12	6	2

Table 1. Assumptions as properties that are known to be correct

DISCUSSION AND CONCLUSION

The ambiguous treatment of assumptions that we found may inhibit students' mathematical thinking. A cause of this ambiguity is the simultaneous usage in the textbooks of 'justification' and 'systematization' as functions of proof (de Villiers, 1990). An implication is that Japanese textbook design might be improved by overcoming the distinction by providing an introductory section on proof structure (Miyazaki, Fujita, and Jones, 2017), including the definition of a proof and sections on 'justification' and 'systematization'. In a section on 'justification', all properties of Group A might be used as assumptions to prove unknown properties. Then, in a section on 'systematization', properties of Group A∩B might be proved by considering the intended local system of properties after stating that the function of proof is different.

References

De Villiers, M. (1990). The role and function of proof on mathematics. *Pythagoras, 24*, 17–24.

Fan, L., Trouche, L., Qi, C., Rezat, S., & Visnovska, J. (Eds.). (2018). *Research on mathematics textbooks and teachers' resources: Advances and issues*. Cham, CH: Springer.

Hanna, G., & De Villiers, M. (2012). Aspects of proof in mathematics education. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: The 19th ICMI Study* (pp. 1–10). New York: Springer.

Miyazaki, M., Fujita, T., & Jones, K. (2017). Students' understanding of the structure of deductive proof, *Educational Studies in Mathematics, 94*(2), 223–239.

A QUALITATIVE STUDY ON THE LEVELS OF MATHEMATICAL EXPLANATION IN JAPANESE ELEMENTARY TEXTBOOK

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INTRODUCTION

In Japan (maybe, also around other countries), one has pointed out that elementary students' ability and skill for explanation in mathematics are lacked. For this reason, we should analyze our own mathematics textbooks because it works as curriculum. (Notes: In Japan, all textbooks are authorized by the Ministry of Education). Our research question is to identify what kind of explanation is intended to teach by the textbooks.

THEORETICAL FRAMEWORK

For our research question, we can assume that there are levels of mathematical explanation, because elementary school has its own educational purpose or intention. Thus, we focus on Miyazaki (1995) who established theoretical framework of the levels of mathematical explanation in elementary school (Table 1). The purpose of Miyazaki (1995) is to improve teaching and learning proof and proving in junior high school students; for this purpose, he also construct the framework because explanation in elementary school will connect with proof. Table 1 shows the contents of the explanation level of the framework. Miyazaki (1995) identify and distinguish 4 levels of explanation in elementary school mathematics; level 1 is lowest, and 4 is highest.

<table>
<tr><th>Perspective
Levels</th><th>Contents of explanation</th><th>Student's thought</th><th>Representation for explanation</th></tr>
<tr><td>1</td><td>It is logical, but not deductive reasoning from universal validated premise for students to the proposition.</td><td rowspan="2">Concrete operation</td><td rowspan="3">Language, figure, and concrete object, but it is excluded that language for representing chain of propositions of numbers and/or figures.</td></tr>
<tr><td>2</td><td rowspan="3">It is logical reasoning from universal validated premise for students to the proposition.</td></tr>
<tr><td>3</td><td rowspan="2">Formal operation</td></tr>
<tr><td>4</td><td>Language for representing chain of propositions of numbers and/or figures.</td></tr>
</table>

Table 1. Levels of explanation in school mathematics. (Miyazaki, 1995, p. 110; translated by author)

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 381–382). Paderborn: Universitätsbibliothek Paderborn.

METHODLOGY

We analyze two pages of textbooks by using the framework. Our intention is not quantitative analysis because we want to get first suggestions or tendency for our purpose; quantitative analysis is next step; here we don't intend to quantitative method or research. Our object is "formula for the area of a parallelogram" in the fifth grade (Shimizu et al., 2015, pp. 124–125), because there are 4 explanations for one problem; since the number of explanations for one problem is most, it is expected that the intention of the textbook appears. Figure 1 is an English textbook with the same contents in Japanese textbook.

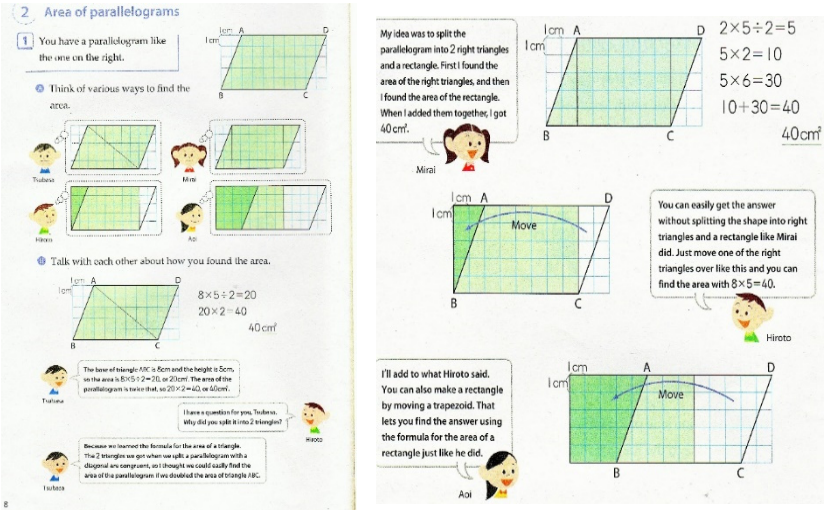

Figure 1. Explanation about validating formula for the area of a parallelogram (Shimizu et al., 2012, pp. 8–9)

RESULT

As a result, we can point out there are only explanation of level 2 or 3 are written in the pages. However, textbook does not say which explanation is better. In other words, we found that textbook does not intend to teach better explanation (at least explicitly). On the other hand, Miyazaki's (1995) framework does not refer how to improve it because it is not his purpose. Thus, we found necessity of another types of theoretical frameworks for improving this kind situations.

Our future task is that, we analyze only one page in this study, so we should analyze more pages and textbooks. In addition, international comparison is necessary for our aim and research community too.

References

Miyazaki, I. (1995). *Research on proof in school mathematics: Though setting the level of explanation at the stage of proof*. Doctoral Thesis in Tsukuba University. (in Japanese)

Shimizu, S., Funakoshi, S., Aoyama, K., Ijima, Y., Ieda, H., Ishida, J., … Watanabe, M. (2012). *Fun with MATH for elementary school* (Grades 5B). Osaka, Keirinkan.

Shimizu, S., Funakoshi, S., Negami, S., Teragaito, S., Aoyama, K., Ijima, Y., … Watanabe, M. (2015). *Wakuwaku Sansu* (Grades 5). Osaka, Keirinkan. (in Japanese)

ANALYZING MATH TEXTBOOKS IN ELEMENTARY SCHOOL MATHEMATICS IN JAPAN: FOCUSING ON THE EXPLANATIONS OF MULTIPLICATION OF DECIMAL NUMBERS

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PURPOSE OF THIS STUDY AND RESEARCH QUESTION

The purpose of this study is to analyze how the multiplication of decimal numbers is explained in two authorized textbooks used in elementary school mathematics in Japan. (Both textbooks are authorized by the Ministry of Education in Japan.) A reason for focusing on the explanation of multiplication of decimal numbers is that multiplications are extended from natural numbers to rational numbers by using the concept of proportion. This extension requires students to reconsider the meaning of multiplications which they learnt in the earlier grades of their elementary school. This is shown as being one of the important steps in elementary school mathematics in Japan. Despite its simplicity of calculation, it is difficult for many students to extend the meaning of multiplication from natural numbers to rational numbers (e.g. Geer, 1994; Tirosh et al., 1989). Therefore, it is significant to clarify what you emphasize in this learning through analysis of decimal multiplication textbooks.

The research question is:

How are the components related to the explanation of Decimal Multiplication in the "Course of Study" as they appear in the school mathematics textbooks?

Theoretical framework and method of analysis

In order to identify the multiplication of decimal numbers is explained in the two authorized textbooks, TIMSS framework, in particular, Robitaille et al. (1993) and Valverde et al. (2002) and the framework specific to the explanation of the multiplication of decimal numbers were adopted in this study. By using TIMSS framework, units of textbooks related to the multiplication of decimal numbers were divided into smaller structures called “blocks”, and each block was coded in terms of “content”, “performance expectation” and “perspectives”. Next, by using the framework specific to the explanation of the multiplication of decimal numbers, these “blocks” and their relationships were analysed. (for similar approaches, see Fujita & Jones, 2014; Jones & Fujita, 2013).

Findings and Discussion

The following results were found from the analysis (the results of the frequency of “contents,” “performance expectation,” and “perspective” will be shown on the presentation poster) with the two textbooks analysed. (According to one textbook

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 383–384). Paderborn: Universitätsbibliothek Paderborn.

published by Tokyo Shoseki in 2014 and and another textbook published by Keirinkan in 2014)

- There are two contents of explanation. One is for the reason of the formulation and an explanation on how to calculate.
- Each explanation is based on the proportional relationship between the two quantities.
- Both analysed textbooks begin by discussing an everyday life context, i.e. purchasing of ribbons and finding prices. This explains multiplication can be given because the relationship between length and price is proportional.
- Both textbooks introducing the multiplications with decimals by extending from a set of natural numbers to a set of rational numbers.
- Depending on the textbook, proportional relationships may be expressed in terms of words and cases where proportional relationships are not expressed.
- Explanations that make use of the real world and explanation of the number world are appear at about the same ratio.

From the above, there are differences in the textbooks in terms of their explanations of multiplications with decimals, although they follow the same Course of Study. Implications to practice will also be provided in the poster.

References

Fujita, T., & Jones, K. (2014). Reasoning-and-proving in geometry in school mathematics textbook in Japan. *International Journal of Educational Research*, *64*, 81–91.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning mathematics* (pp. 61–85). New York: State University of New York Press.

Jones K., & Fujita, T. (2013). Interpretations of national curricula: The case of geometry in textbooks from England and Japan. *ZDM Mathematics Education*, *45*(5), 671–683.

Robitaille, D. F., Schmidt, W. H., Raizen, S., McKnight, C., Britton, E., & Nicol, C. (1993). *Curriculum frameworks for mathematics and science* (TIMSS Monograph No. 1). Vancouver: Pacific Educational Press.

Tirosh, D., & N. Almog. (1989). Conceptual adjustments in progressing from real to complex numbers. *In* Vergnaud, J., Rogalski, & M. Artigue (Eds.), *Proceedings of the thirteenth annual conference of the International Group for the Psychology of Mathematics Education*, (Vol. 3, pp. 221–227). Paris: G. R. Didactique.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer.

A STUDY OF THE CHARACTERISTIC OF JAPANESE TEXTBOOKS TOWARDS THE GROWTH OF CHILDREN'S MATHEMATICAL REPRESENTATION

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INTRODUCTION

In Japan, we are focusing on the growth of representation skills, because Japanese children have weaker thinking and representation skills than children in other countries. Thus, teachers require children to be able to write the "what", "how" and "why" and utilize various representations. The purpose of this research is to clarify what kinds of measures are taken up in elementary and junior high school textbooks in Japan for the growth of representation skills. Representation skills in this paper refer to logically writing these concepts shown above. The method of research examines the representation skills in the Japanese National Curriculum. Next, based on the textbooks of the two companies, the descriptions of the textbooks of means to foster representation skills are extracted, and the characteristics are compared for each grade.

MATHEMATICAL REPRESENTATIONS

A summary and discussion were conducted on the kinds of mathematical representations required as well as the process of its development based on the national curriculum and the national achievement test. Both of the curriculum and test possess an explicit process of the continuity between elementary school and junior high school. As grade level increases, subject matter changes from arithmetic to mathematics as mathematical representations become more formal and mathematical, as well as more logical and systematic. More concretely, as academic level gradually increases, so do the activities requiring more logical representations, for instance, using facts, methods, and reason. These activities gradually transition from usage extracted from everyday situations to those that are utilized in mathematical contexts. Further, similar activities are implemented in multiple grade levels.

MEASURES IN JAPANESE TEXTBOOKS

Measures used by the textbooks were also organized and considered in this study. As students progress through grade levels, their understanding and usage of mathematical representations, in the same manner as their textbooks, becoming more formal, for instance, from individual mathematical representation to more formal representation, from Manipulative Representation and Illustrative Representation to more Linguistic Representation, as well as the usage of more Symbolic Representation of mathematical formula. In the third year of elementary school, students are required to use the logical words ‘first’ and ‘next’ and for argument and reason use ‘I think’ and ‘because of reason A’. In the fourth and fifth grades of elementary school, students are required to

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 385–386). Paderborn: Universitätsbibliothek Paderborn.

clarify what they want to use before writing and to use logical representations when comparing things. In the sixth grade of elementary school and the first year of junior high school, students are required to use mathematical representations more sophisticatedly based on learning content. In the second year of junior high school, students are urged to represent matter based on other perspectives, and in the third grade to represent matters more logically. Additionally, measures of mathematical representations are conducted not only through a textbook's speech bubbles or comments on the subject matter, but also by preparation and usage of mathematical notes and reports that guide students in the development of their understanding and usage of mathematical representations. Certain textbook companies include content to develop students' understanding and usage of mathematical representations by including content focused on knowledge and usage of vocabulary necessary for students to represent themselves logically. These scaffolds show the logical relationship of content, for example, words such as "first", "next", "then", "so", "for example", "if A is, it is B". Also, similar activities are provided in multiple grades here as well. The two textbooks company who analyzed in this study had different methods of providing this type of content in the same grade. Each textbook is designed within the scope of the national curriculum and the national achievement test. For the same degree of mathematical representations mandated in each grade level by the national curriculum, the learning content in each textbook is organized differently in each grade level.

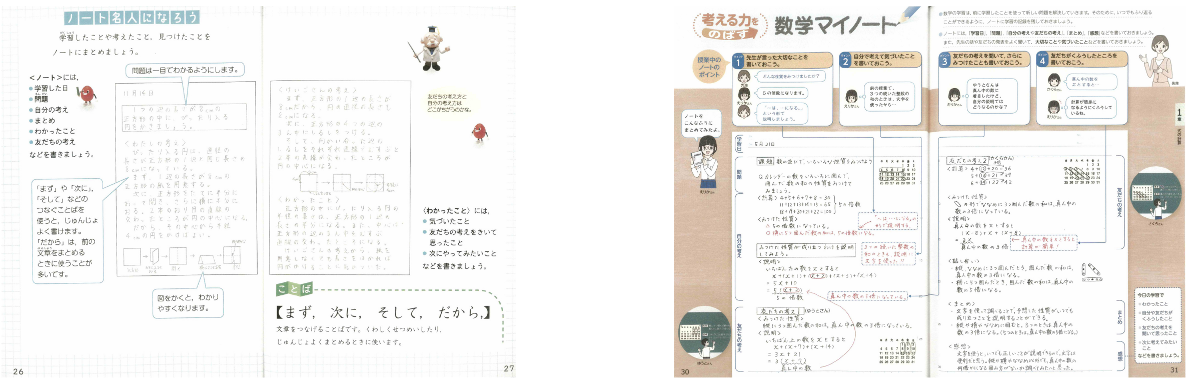

Figure 1: Mathematical Notes on a Textbook (Elementary School & Junior High School)

FINAL REMARKS

Japanese elementary and junior high school textbooks use various techniques in each grade level to scaffold the development of students' understanding and usage of mathematical representation, which will become more prominent in the future.

References

Ichimatu, S., & Okada, Y. (2015). *Learn with everyone for elementary school arithmetic (3rd grade)*, Gakkotosyosyuppann. (in Japanese)

Fujii, T., & Yasuno, H. (2015). *New edition new mathematics 2*, Tokyosyoseki. (in Japanese)

A STUDY ON THE SENIOR HIGH SCHOOL STUDENTS' ALGEBRAIC ACTIVITIES AND THE DIGITAL TOOLS IN JAPANESE DIGITAL MATHEMATICS TEXTBOOK

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Naruto University of Education, Japan

THEORETICAL FRAMEWORK AND METHODOLOGY

Students' algebraic notions and skills are very important for purpose of mathematics education, however it is not taught enough. In this study, we analyze Japanese digital mathematics textbook in senior high school with focusing on students' algebraic skills or notions, because it may develop new didactical materials. The activities to develop students' algebraic notion and skills need the suitable tools for it. So, we focus on not students but materials used by student. Here, we have two research questions:

RQI) What theoretical framework is normative for developing students' algebraic notions and skills?

RQII) From this theoretical framework, what are mathematics textbooks insufficient?

For learning algebra, generalization plays essential role; especially, Dörfler (1991) pointed out that to use algebraic letters relates with students' activities. For instance, according to Fujii (2000), when students can not consider about relationships between mathematical objects, they can not use algebraic letters well; Dörfler (1991) said students abstract these relationships through their own activities. Thus, our theoretical framework is based on Dörflers' (1991) theory of generalization that to use algebraic letters relates with students' activities, and methodology is qualitative analyses of the digital textbook; we focus on unit of quadratic function because it is compulsory and core unit of learning algebra in senior high school. We compare and analyze the digital and non-digital textbooks "Mathematics I" of Suken Shuppan (Oshima et al., 2011a, 2011b); the subject "Mathematics I" is the only compulsory in Japanese mathematics curriculum. In addition, it is the most adopted textbook in Japanese senior high school (42.4%; cf. Zizitsushin, 2014).

RESULT OF ANALYSIS

As a result, in the situation introducing vertex form of quadratic function, we can point out the layout of digital textbook is identical with non-digital textbooks; because there is social constraint that if they are identical, digital textbooks are not need to authorized by the Ministry of Education. On the other hand, each problem has an icon to access the digital tools. When students click the icon on the right of explanation, they can use digital tool (Figure 1). As shown, it is the graph that is changed by the value of p or q that is inputted by students. From our theoretical view, almost of all these tools are for *validating* (and/or *justifying*) explanation by sentences. In other words, we found that

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 387–388). Paderborn: Universitätsbibliothek Paderborn.

these tools do not allow students' activities (in sense of Dörfler (1991)). Based on this result, we discuss nature of digital tools and propose concrete tools and theoretical perspective for digital tool which realize Dörfler's (1991) activities.

We have two future tasks. First, we have to consider what tools are better for activities to develop students' algebraic notions and skills, expand this theory to besides unit of quadratic function and refer to other studies about digital textbook. Second, this study is a case study in Japan, thus, we should analyze other country's (digital) textbooks.

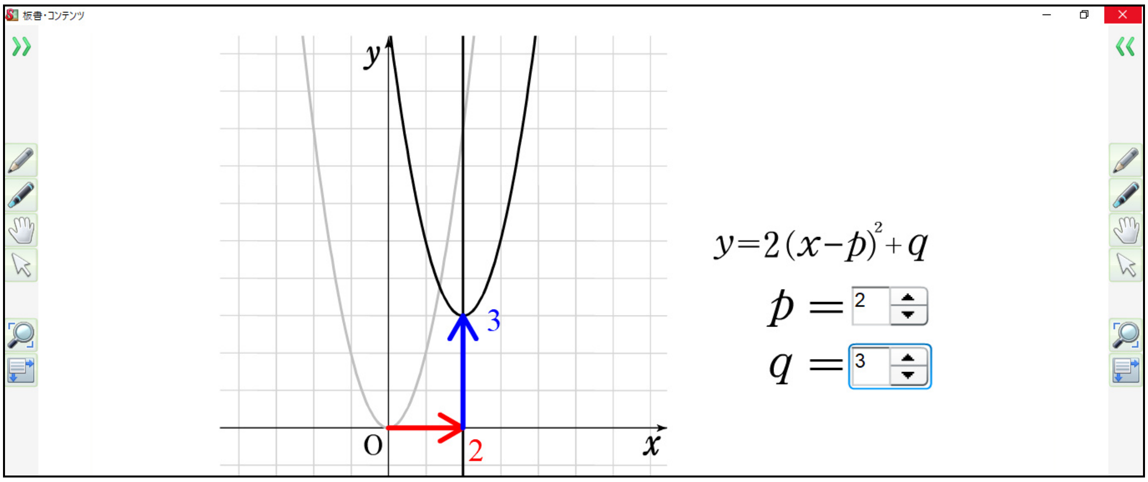

Figure 1. Example of digital tool in the digital textbook (Oshima et al., 2011b, p. 74)

References

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. J. Bishop, S. Mellin-Olsen, & J. v. Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 63–85). Dordrecht, Netherlands: Kluwer.

Fujii, T. (2000). About difficulty of "representating the equation". In *Proceedings of Conference of Japan Society of Mathematical Education*, 33, (pp. 349–354). Japan Society of Mathematical Education. (in Japanese)

Oshima, T., Tsuboi, T., Kakei, S., Hattori, T., Kato, H., Fukaya, K, ... Yoshida, S. (2011a). *A revised edition Mathematics I*. Suken Shuppan. (in Japanese)

Oshima, T., Tsuboi, T., Kakei, S., Hattori, T., Kato, H., Fukaya, K, ...Yoshida, S. (2011b). *CoNETS The digital textbook for leaners: A revised edition Mathematics I for Windows*. Suken Shuppan. (in Japanese)

Zizitsushin. (2014). *Status of adoption of elementary school textbook for 2013: Report of Ministry of Education*, Naigaikyoiku, Zizitsushinsha. (in Japanese)

EXPLORATORY RESEARCH ON UNDERSTANDING PROOF BY COINCIDENCE: BUILDING AN ANALYTICAL FRAMEWORK BY COMPARING JAPANESE TEXTBOOKS

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INTRODUCTION

"Proof by coincidence" is a type of indirect proof defined as follows:

> Coincidence method consists in constructing a geometric figure which possesses all the properties in question and then showing that it coincides with the given geometric figure (Shute, Shirk, Porter, & Irwin, 1949, p. 68).

While there is an opinion that proof by coincidence is not an indirect proof (cf. Byham, 1969), it is usually treated as such in Japan (The Japan Society of Mathematical Education, 2013). The seven standard 9th-grade mathematics textbooks in Japan prove the Converse of the Pythagorean Theorem (CPT) by this method, although three textbooks apply it only in a single case. In general, research on indirect proof tends to focus only on proof by contradiction and proof by contrapositive (e.g., Antonini & Mariotti, 2008). Therefore, when researching indirect proofs, we should pay more attention to proof by coincidence.

The purpose of this paper is to build an analytical framework for empirical studies seeking to understand proof by coincidence. Due to a dearth of scientific literature regarding how to teach this topic, it is likely that textbooks have a high impact on its teaching practices, relative to other topics. To hypothesize the potential problems of teaching proof by coincidence, we will investigate the variety of ways in which the Pythagorean Theorem (PT) and the CPT are presented in Japanese textbooks.

METHOD AND RESULT

We examined all seven junior high school mathematics textbooks, focusing on the differences among the figures and the centered content. Table 1 shows a comparison of the figures for the PT and the CPT. Publishers *A*, *B*, and *C* use the same figure for both theorems, while the other four use different figures. The presence or absence of a right-angle mark is the key difference between Type I and Types II/III.

Table 2 shows the centered content used to present the CPT. When presenting the PT, all the textbooks center $a^2+b^2=c^2$, but they differ in their presentation of the CPT. Publishers *A*, *E*, *F*, and *G* center the antecedent $a^2+b^2=c^2$, which indicates that they consider the emphatic points of the PT and the CPT to be the same.

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 389–390). Paderborn: Universitätsbibliothek Paderborn.

Type	PT	CPT	Publisher
F1	Type I	Type I	*A*, *B*, *C*[*1]
F2	Type I	Type II	*D*, *F*, *G*
F3	Type I	Type III	*E*

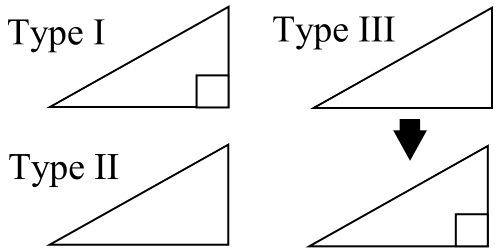

Table 1. Comparison between the PT and the CPT figures ([*1] Publisher *C* uses Type I for the CPT but with a different color.)

Type	Centered contents	Publisher
C1	$a^2+b^2=c^2$	*A*, *E*, *F*, *G*
C2	Nothing	*B*, *D*
C3	if $a^2+b^2=c^2$, then $\angle C = 90°$	*C*

Table 2. Centered contents of the CPT

DISCUSSION

When we prove the CPT by coincidence, we must refer to the PT in the proof. As this can be very confusing for students, an effort should be made to resolve this confusion. However, as shown in Tables 1 and 2, the figures and centered contents in some textbooks do not differ for the PT and the CPT (Types F1 and C1). It is noteworthy that we are unsure whether the differences in the ways that textbooks present the PT and the CPT can contribute to resolving students' confusion. Therefore, empirical research on students' process of understanding proof by coincidence should be conducted from this point of view. Thus, we raise the following question: what influence does a teacher's way of summarizing the process of proving by coincidence have on students' understanding?

Acknowledgements

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References

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM Mathematics Education*, *40*(3), 401–412.

Byham, F. (1969). *Indirect proof in geometry from Euclid to the present* (Electronic Dissertation). Retrieved from https://etd.ohiolink.edu/

Shute, W. G., Shirk, W. W., Porter, G. F., & Irwin, M. E. (1949). *Plane geometry*. American Book Company.

The Japan Society of Mathematical Education (Ed.). (2013). Method of indirect proof. In *Japanese-English/English-Japanese Dictionary for Practical School Mathematics Usage* (p. 418). Tokyo: Toyokan Publishing.

A MATHEMATICS COLLEGE TEXTBOOK: DESIGN INTENTIONS AND READING PRACTICE

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RATIONALE FOR RESEARCH

In mathematics, textbooks are an integral component of the classroom, providing immediate access to a wealth of knowledge (Fan, Zhu, & Miao, 2013) and in theory enabling development of mathematical readings skills. In research, mathematical reading has been recognised as difficult due to complexities of mathematical texts: text is often dense, contains abstract concepts, and uses new language or familiar language semantically differently (Avalos, Bengochea, & Secada, 2015). Textbooks could support students' practice of these reading skills, extending the textbook's role beyond providing lesson material.

Empirical research on textbooks has recently increased but more is required for the field to progress and for findings to translate into improving this potentially powerful resource (Fan, 2013). In addition, research into how students use their textbooks is also lacking (Fan et al., 2013) despite students comprising the main audience.

THEORETICAL FRAMEWORK

To address this paucity of research, we adopted Weinberg and Wiesner's (2011) adaptation of reader-oriented theory for mathematics textbooks. Weinberg and Wiesner identified three primary readers of a mathematics textbook: the reader within a textbook author's mind (Intended), the actual reader of the text (Empirical) and the reader constructed by the text (Implied). The framework focusses on how readers use a textbook at the micro-level, identifying individual design elements of the text, such as expository text, worked examples or key points. We investigated the Intended and the Empirical Reader and justified our interpretation of the text using the Implied Reader.

EXPLICATION OF METHODS

We used mixed methods to separately investigate the Intended Reader and then the Empirical Reader. To investigate the Intended Reader, we interviewed three authors of a textbook, asking them how they expected their textbook to be used and thematically analysing the resulting transcripts to explore their different opinions. To investigate the Empirical Reader, we used eye-tracking technology to record how 31 students read an extract from the same textbook. We analysed the data to compare differences in reading time and processing depth between design elements. In addition, student participants were split into groups of mathematics and psychology students to explore the influence of mathematical expertise on reading practice. Findings of both

In Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development* (pp. 391–392). Paderborn: Universitätsbibliothek Paderborn.

investigations were compared to explore congruence between the Intended Reader and Empirical Reader.

STATEMENT OF RESULTS

1. Authors tended to accurately predict how students would read, for instance predicting that students would not read passages of expository text; students did read these but did not process them as deeply as other design elements.
2. Authors reported that editors are primarily in charge of textbook design and that authors have the editor's instructions in mind rather than a prospective reader.
3. For students, there were significant differences between both time spent and depth of engagement for different design elements, but these differences were not consistent between the two measures.
4. A post-test of learning from the textbook showed a significant difference of ability between mathematics and psychology students but there were no significant between-group differences in reading practice.

IMPLICATIONS FOR FUTURE RESEARCH

Significant differences between how design elements were read might be unsurprising given that different elements are designed for different purposes. However, investigating reading for each element could help understand the cognitive processes each requires, which could help to determine their actual learning benefit. Further research could also investigate relative expert/novice textbook reading behaviour. Authors did not suggest that elements were designed with expertise in mind, but it is possible that students with greater knowledge or experience make better or different use of different elements. More broadly, the reality is that textbooks are designed to sell, an aim not necessarily synonymous with learning benefit. However, I propose that future research could synthesise these two aims by using empirical research to help everyone understand how textbooks can become more effective learning resources.

References

Avalos, M. A., Bengochea, A., & Secada, W. G. (2015). Reading mathematics: More than words and clauses; more than numbers and symbols on a page. In K. L. Santi & D. K. Reed (Eds.), *Improving reading comprehension of middle and high school students* (pp. 49–74). New York: Springer.

Fan, L. (2013). Textbook research as scientific research: Towards a common ground on issues and methods of research on mathematics textbooks. *ZDM Mathematics Education, 45*(5), 765–777.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education, 45*(5), 633–646.

Weinberg, A., & Wiesner, E. (2011). Understanding mathematics textbooks through reader-oriented theory. *Educational Studies in Mathematics, 76*, 49–63.